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THE QUARTERLY JOURNAL OF
MATHEMATICS
(OXFORD SERIES)

Volume 8 *No.* 30 *June* 1937

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OXFORD
AT THE CLARENDON PRESS
1937

Price 7s. 6d. net

PRINTED IN GREAT BRITAIN BY JOHN JOHNSON AT THE OXFORD UNIVERSITY PRESS
Entered as second class at the New York U.S.A., Post Office

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Edited by T. W. CHAUNDY, J. HODGKINSON, E. G. C. POOLE
With the co-operation of A. L. DIXON, E. B. ELLIOTT, W. L. FERRAR, G. H. HARDY, A. E. H. LOVE, E. A. MILNE, E. C. TITCHMARSH

THE QUARTERLY JOURNAL OF MATHEMATICS (OXFORD SERIES) is published in March, June, September, and December, at a price of 7s. 6d. net for a single number with an annual subscription (for four numbers) of 27s. 6d. post free.

Papers, of a length normally not exceeding 20 printed pages of the Journal, are invited on subjects of Pure and Applied Mathematics, and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. Contributions can be accepted in French and German, if in typescript (formulae excepted). Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints. Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher (Humphrey Milford, Oxford University Press, Amen House, Warwick Square, London, E.C. 4).

HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
AMEN HOUSE, LONDON, E.C.4

ON THE REPRESENTATION OF A QUADRATIC FORM AS A SUM OF SQUARES OF LINEAR FORMS

By CHAO KO (*Manchester*)

[Received 13 October 1936]

1. LET $$f(x_1,\ldots,x_n)=\sum_{i,j=1}^{n}a_{ij}x_ix_j \quad (a_{ij}=a_{ji}),$$ where a_{ij} are given integers, be a positive-definite form. There are many interesting problems connected with the representation of $f(x_1,\ldots,x_n)$ in the form

$$f(x_1,\ldots,x_n)=\sum_{i=1}^{r_n}(b_{i1}x_1+\ldots+b_{in}x_n)^2, \tag{1}$$

where the b's are numbers of a particular kind. We may ask if such a representation is possible, or again what is the smallest value of r_n, say R_n, such that every definite form $f(x_1,\ldots,x_n)$ can be so represented.

Thus, if the b's are rational numbers, Landau* first proved that for $n=2$, a representation (1) exists when $r_2=5$; and Mordell† proved that, for any given n, a representation (1) exists when $r_n=n+3$, and proved that $R_2=5$, leaving unsettled the value of R_n when $n>2$.

Suppose next that the b's are integers. Mordell proved‡ that $R_2=5$, and conjectured§ that $R_n=n+3$. The well-known theorem that every positive integer is the sum of four integer squares can be stated as $R_1=4$.

The other problem is that of representation for given n and given $r_n<R_n$. Mordell has given‖ the conditions for $n=2$, $r_2=2, 3, 4$, these being the same whether the b's are rational numbers or integers. I shall give in § 3 a new proof for his theorem when $n=2$, $r_2=4$.

All these problems can be still further generalized by replacing (1) by the representation**

$$f(x_1,\ldots,x_n)=\sum_{i=1}^{s_n}m_i(b_{i1}x_1+\ldots+b_{in}x_n)^2, \tag{2}$$

* Landau, *Archiv der Math. und Physik* (3), 7 (1904), 271–7.

† Mordell (1), *Math. Zeits.* 35 (1932), 1–15.

‡ Mordell (2), *Quart. J. of Math.* (Oxford) 1 (1930), 276–88.

§ Ibid. 282. See also § 6.

‖ Mordell (1). Mordell (2), 279. Mordell, *J. für Math.* 167 (1931), 12–19. Recently he found a new proof for the case $n=2$, $r_2=4$ by using quaternions.

** Mordell (2), 288.

where the m_i are rational numbers or integers, and $f(x_1,...,x_n)$ is a general quadratic form i.e. not necessary positive-definite. For $n = 2$, the b's integers, and $s_2 = 2, 3, 4, 5$, I gave* the necessary and sufficient conditions for the representation (2) to exist for each of the possible combinations $m_i = \pm 1$.

When the b's are rational, I have found a short proof for the theorem $R_n \leqslant n+3$ by using the result $R_2 = 5$. When the b's are integers, I prove that $R_5 = 8$ (and consequently $R_4 = 7$, $R_3 = 6$, $R_2 = 5$) by using a suggestion from Professor Mordell about the method of attack. The proof depends upon expressing $f(x_1,..., x_5)$ as the part arising from a definite properly-primitive quadratic form $f(x_1,..., x_8)$ with determinant unity on putting $x_6 = x_7 = x_8 = 0$. Since there is only one class of properly-primitive definite forms in eight variables with determinant unity (as will be proved in § 4), $f(x_1,..., x_8)$ can be written as a sum of eight linear squares, and hence the result.

2. I now prove that $R_n \leqslant n+3$, where the b's are rational. For $n = 3$, by Meyer's theorem,†

$$f(x_1, x_2, x_3) = u^2+v^2$$

is always solvable in integers x_1, x_2, x_3, u, v. Hence, by applying a unitary transformation, we can suppose $a_{11} = a_1^2+a_2^2$. Then

$$(a_1^2+a_2^2)f(x_1, x_2, x_3) = (a_{11}x_1+a_{12}x_2+a_{13}x_3)^2+\phi(x_2, x_3),$$

where $\phi(x_2, x_3)$ is a positive-definite quadratic form and so is a sum of five linear squares, since $R_2 = 5$. Hence

$$(a_1^2+a_2^2)f(x_1, x_2, x_3) = \sum_{i=1}^{6} L_i^2,$$

where the L's are linear forms in x_1, x_2, x_3, and so

$$f(x_1, x_2, x_3) = (a_1 L_1+a_2 L_2)^2/a_{11}^2+(a_1 L_2-a_2 L_1)^2/a_{11}^2+...+ \\ +(a_1 L_5+a_2 L_6)^2/a_{11}^2+(a_1 L_6-a_2 L_5)^2/a_{11}^2.$$

For $n > 3$, by Meyer's theorem,

$$f(x_1,..., x_n) = u^2$$

is solvable in integers and so we can assume $a_{11} = a^2$. Then

$$f(x_1,..., x_n) = [(a_{11}x_1+a_{12}x_2+...+a_{1n}x_n)^2+\phi(x_2,..., x_n)]/a^2,$$

* Ko, *Proc. London Math. Soc.* 42 (1936), 171–85.

† Mordell, *J. für Math.* 164 (1931), 49, or Bachmann (1), *Zahlentheorie*, vol. 4, part 1, 266. I have also found a new proof by expressing the indefinite form in five variables as a part of an indefinite form in seven variables with determinant unity.

where ϕ is a positive-definite quadratic form in $n-1$ variables. Hence by induction we obtain the result.

3. I illustrate my method by proving the theorem due to Mordell.*

THEOREM 1. *A representation of the positive-definite form $f(x_1, x_2)$ as*

$$f(x_1, x_2) = (b_{11}x_1+b_{12}x_2)^2+\ldots+(b_{41}x_1+b_{42}x_2)^2, \tag{3}$$

where the b's are integers, exists if and only if

$$D_2 = a_{11}a_{22}-a_{12}^2 \neq 4^\rho(8\tau-1),$$

where $\rho\ (\geqslant 0)$, $\tau\ (>0)$ are integers.

The condition is necessary since Mordell has shown that, if a representation (3) exists, we have

$$a_{11} = \sum_{i=1}^{4} b_{i1}^2, \qquad a_{12} = \sum_{i=1}^{4} b_{i1}b_{i2}, \qquad a_{22} = \sum_{i=1}^{4} b_{i2}^2,$$

and

$$\begin{aligned} D_2 &= \Big(\sum_{i=1}^{4} b_{i1}^2\Big)\Big(\sum_{i=1}^{4} b_{i2}^2\Big)-\Big(\sum_{i=1}^{4} b_{i1}b_{i2}\Big)^2 \\ &= (b_{11}b_{22}-b_{21}b_{12}-b_{31}b_{42}-b_{41}b_{32})^2+ \\ &+(b_{11}b_{42}-b_{21}b_{32}+b_{31}b_{22}-b_{41}b_{12})^2+(b_{11}b_{32}+b_{21}b_{42}-b_{31}b_{12}-b_{41}b_{22})^2, \end{aligned}$$

i.e. D_2 is a sum of three squares and hence $D_2 \neq 4^\rho(8\tau-1)$.

It is enough to prove the sufficiency of the theorem for primitive forms, i.e. $(a_{11}, a_{12}, a_{22}) = 1$. For if

$$(a_{11}, a_{12}, a_{22}) = \delta = \sum_{i=1}^{4} \delta_i^2,$$

and

$$f(x_1, x_2)/\delta = \sum_{i=1}^{4} L_i'^2,$$

where the L''s are linear forms, then, by Euler's theorem,

$$f(x_1, x_2) = \sum_{i=1}^{4} \delta_i^2 \sum_{i=1}^{4} L_i'^2$$

can be expressed as a sum of four linear squares. Also $D_2 = \delta^2\Delta_2$, and so $D_2 = 4^\rho(8\tau-1)$ if and only if Δ has the same form.

I prove the sufficiency of the condition by using

LEMMA 1. *Let* $f(x_1, x_2) = a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2$ *be a positive-definite primitive form. If $D_2 \not\equiv 0, 4, 7 \pmod 8$, then there exist integers a_{33}, a_{34}, a_{44} such that the quaternary form*

$$g_4 = f(x_1, x_2)+2x_1x_4+a_{33}x_3^2+2a_{34}x_3x_4+a_{44}x_4^2$$

is a positive-definite form with determinant unity.

* Mordell (2), 279.

For this means that

$$\begin{vmatrix} a_{11} & a_{12} & 0 & 1 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 1 & 0 & a_{34} & a_{44} \end{vmatrix} = 1,$$

whence
$$a_{34}^2 D_2 + 1 = a_{33}(a_{44} D_2 - a_{22}). \tag{4}$$

Since the form $f(x_1, x_2)$ is primitive, we can, on replacing it by an equivalent form, suppose $(a_{22}, D_2) = 1$*. Then, by Dirichlet's theorem,† there exist for variable integers a_{44} infinitely many primes p or $2p$ of the form $a_{44} D_2 - a_{22}$. We show that (4) is solvable by selecting a_{44} so that $a_{44} D_2 - a_{22} = p$ or $2p$, where p is an odd positive prime, and $(-D_2/p) = 1$, the symbol being that of quadratic residuacity.

Suppose first $D_2 \equiv 1 \pmod 4$. Take $a_{44} D_2 - a_{22} = p$. Then

$$(-D_2/p) = (-1)^{\frac{1}{2}(p-1)}(p/D_2) = (-1)^{\frac{1}{2}(p-1)}(-a_{22}/D_2).$$

Since $a_{44} \equiv 1 + a_{22}$ or $3 + a_{22} \pmod 4$ make $p \equiv 1$ or $3 \pmod 4$ respectively, we can always make $(-D_2/p) = 1$ by suitable choice of a_{44}.

Suppose next $D_2 \equiv 3 \pmod 8$. If $(-a_{22}/D_2) = 1$, take $a_{44} D_2 - a_{22} = p$, and we have $(-D_2/p) = (-a_{22}/D_2) = 1$. If $(-a_{22}/D_2) = -1$, we take $a_{44} D_2 - a_{22} = 2p$; then $(2/D_2) = -1$ and so

$$(-D_2/p) = -(p/D_2)(2/D_2) = -(2p/D_2) = -(-a_{22}/D_2) = 1.$$

Hence we can always make $(-D_2/p) = 1$.

Finally, suppose $D_2 = 2D_2'$, $D_2' \equiv 1 \pmod 2$. Setting

$$a_{44} D_2 - a_{22} = p,$$

we have
$$(-D_2/p) = (2/p)(-D_2'/p).$$

But
$$(2/p) = (-1)^{(p^2-1)/8},$$

$$(-D_2'/p) = (-1)^{\frac{1}{2}(p-1)\frac{1}{2}(D_2'+1)}(p/D_2') = (-1)^{\frac{1}{2}(p-1)\frac{1}{2}(D_2'+1)}(-a_{22}/D_2'),$$

and so
$$(-D_2/p) = (-1)^{(p^2-1)/8+(p-1)(D_2'+1)/4}(-a_{22}/D_2').$$

Choose $p \equiv 1 \pmod 8$, when $(-a_{22}/D_2') = 1$; and $p \equiv 5 \pmod 8$, when $(-a_{22}/D_2') = -1$. Then always $(-D_2/p) = 1$. Hence (4) is solvable in a_{33}, a_{34}, a_{44}.

* Dickson, *Studies in the theory of numbers*, 8, § 5.

† Landau (1), *Vorlesungen über Zahlentheorie*, Satz 155.

Denote by D_1, D_2,..., D_n the determinants of $f(x)$, $f(x_1, x_2)$,..., $f(x_1,..., x_n)$, where $f(x_1,..., x_i)$ is obtained by putting

$$x_{i+1} = ... = x_n = 0$$

in $f(x_1,..., x_n)$. It is known that

$$f(x_1,..., x_n) = D_1 X_1^2 + D_2 X_2^2/D_1 + ... + D_n X_n^2/D_{n-1},$$

where $X_1,..., X_n$ are rational linear forms in $x_1,..., x_n$, and so $f(x_1,..., x_n)$ is definite and positive if $D_1 > 0,..., D_n > 0$.

Now $D_1 = a_{11}$, $D_2 = a_{11}a_{22} - a_{12}^2$, $D_3 = a_{33} D_2$, $D_4 = 1$ in g_4. Hence g_4 is positive-definite if $a_{33} > 0$, for, by hypothesis, $f(x_1, x_2)$ is positive-definite and so $a_{11} > 0$, $a_{11}a_{22} - a_{12}^2 > 0$. Since p or $2p$ is positive, it is clear from (4) that $a_{33} > 0$. Hence the lemma is established.

Now we can prove the sufficiency of the condition of Theorem 1. For, by Lemma 1, if $D_2 \not\equiv 0, 4, 7 \pmod 8$, there exist integers a_{33}, a_{34}, a_{44}, such that g_4 is a positive-definite form with determinant unity. Since the class number of a positive-definite quadratic form in four variables with determinant unity is one,* we have

$$f(x_1, x_2) + 2x_1 x_4 + a_{33} x_3^2 + 2a_{34} x_3 x_4 + a_{44} x_4^2 = \sum_{i=1}^{4} L_i''^2,$$

where the L'''s are linear forms in x_1, x_2, x_3, x_4. Putting $x_3 = x_4 = 0$, we obtain a representation (3) of the form $f(x_1, x_2)$.

The case $D_2 \equiv 0 \pmod 4$ but $D_2 \neq 4^\rho(8\tau - 1)$ can be reduced to the previous one. For then a_{22} is odd, since we can assume $(D_2, a_{22}) = 1$. By replacing x_2 by $x_2 + \alpha x_1$, we can make a_{11} even, and so a_{12} is even. If $a_{11} \not\equiv 0 \pmod 4$, we should have $D_2 \not\equiv 0 \pmod 4$. Hence $a_{11} \equiv 0 \pmod 4$, and the transformation $2x_1 = y_1$, $x_2 = y_2$ carries f into a primitive form with determinant $D_2/4 \neq 4^\rho(8\tau - 1)$.

4. The proof of $R_5 = 8$ requires the following lemmas:

LEMMA 2. *All the properly-primitive quadratic forms in eight variables with determinant unity are equivalent to* $\sum_{i=1}^{8} x_i^2$.†

The minimum of a quadratic form f_8 in eight variables with determinant unity is 1 or 2.‡ If f_8 represents 1, then $f_8 \sim x_1^2 + \bar{f}_7$, where $\bar{f}_7$ is

* Bachmann (1), vol. 4, part 2, 355–7.

† At the same time Mordell proved that there are exactly two classes of forms, one properly-primitive and the other improperly-primitive. His paper 'The definite quadratic forms in eight variables with determinant unity' may appear in due course in the *Journal de Math.*

‡ Blichfeldt, *Math. Zeits.* 39 (1934), 1–15.

a form in seven variables with determinant unity. Hence* $f_7 \sim \sum_{i=1}^{7} x_i^2$ and our theorem is true in this case.

Suppose f_8 does not represent 1. Let the adjoint form of f_8 be

$$F_8 = \sum_{i,j=1}^{8} A_{ij} x_i x_j.$$

Then we can find a form $f_8' \sim f_8$ such that, in F_8',

$$A_{11}' \leqslant 2.$$

Hence $$f_8 \sim a_{11} x_1^2 + 2 \sum_{i=2}^{8} a_{1i} x_1 x_i + f_7(x_2, \ldots, x_8),$$

where f_7 is a form in seven variables with determinant $D_7 = 1$ or 2. If $D_7 = 1$, f_7 represents 1 and hence f_8 represents 1, contrary to hypothesis. Thus $D_7 = 2$. Let the adjoint form of f_7 be

$$F_7 = \sum_{i,j=2}^{8} B_{ij} x_i x_j.$$

Then as above, we have $f_7' \sim f_7$ with

$$B_{22}' \leqslant \sqrt[7]{(64^2)} < 4,$$

and so $$f_7 \sim b_{22} x_2^2 + 2 \sum_{i=3}^{8} b_{2i} x_2 x_i + f_6(x_3, \ldots, x_8),$$

where the determinant of f_6 is $D_6 = 1$, 2, or 3. If $D_6 = 1$ or 2, and

$$f_6 = \sum_{i,j=3}^{8} c_{ij} x_i x_j,$$

we have $$c_{33} \leqslant \sqrt[6]{(2 \times 64/3)} = 2\sqrt[6]{(2/3)} < 2,$$

and so f_6 represents 1 or f_8 represents 1, contrary to hypothesis. Hence $D_6 = 3$.

It has been shown recently† that there is only one class of forms in six variables with determinant 3 and not representing 1. We may choose its representative to be the form

$$g_6(x_3, \ldots, x_8) = 2(x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2x_7^2 + x_8^2 + x_3 x_4 + x_4 x_5 + x_5 x_6 + 2x_6 x_7 + x_7 x_8).$$

Hence we can write

$$f_7 \sim b_{22} x_2^2 + 2 \sum_{i=3}^{8} b_{2i} x_2 x_i + g_6(x_3, \ldots, x_8).$$

* Hermite, *Œuvres*, vol. 1, 129.

† Hofreiter, *Monats. für Math. und Physik*, 40 (1933), 129–52.

If we replace x_6 by $x_6 - b_{27}x_2$, we can drop the term x_2x_7 without affecting the form g_6. Continuing this process, we have

$$f_7 \sim g_7 = b_{22}x_2^2 + 2b_{23}x_2x_3 + 2\epsilon x_2x_7 + g_6,$$

where $\epsilon = 0$ or 1, and the determinant of the form is

$$\begin{vmatrix} b_{22} & b_{23} & 0 & 0 & 0 & \epsilon & 0 \\ b_{23} & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 \\ \epsilon & 0 & 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{vmatrix} = 2,$$

i.e.

$$3b_{22} - 4b_{23}^2 - 8\epsilon b_{23} - 10\epsilon^2 = 2.$$

Hence b_{22} is even, say $b_{22} = 2k$.

As above, we have

$$f_8 \sim a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2\epsilon_1 x_1x_7 + g_7 \quad (\epsilon_1 = 0 \text{ or } 1),$$

whose determinant is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & \epsilon_1 & 0 \\ a_{12} & 2k & b_{23} & 0 & 0 & 0 & \epsilon & 0 \\ a_{13} & b_{23} & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 \\ \epsilon_1 & \epsilon & 0 & 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{vmatrix} = 1,$$

i.e.

$$2a_{11} - 3a_{12}^2 + 4(2a_{12}a_{13}b_{23} + 2\epsilon_1 a_{12}b_{23} + 2\epsilon a_{12}a_{13} + 5\epsilon\epsilon_1 a_{12} - \\ - 2ka_{13}^2 - 4k\epsilon_1 a_{13} + 2\epsilon^2 a_{13}^2 - 4\epsilon\epsilon_1 a_{13}b_{23} - 5k\epsilon_1^2 + 2\epsilon_1^2 b_{23}^2) = 1.$$

Hence a_{12} is odd and a_{11} is even, i.e. f_8 is improperly-primitive, and so the lemma is proved.

5. LEMMA 3. *If* $(a, p) = (b, p) = 1$, *the congruence*

$$ax_1^2 + bx_2^2 + c \equiv 0 \pmod{p},$$

where p *is a prime, is always solvable.**

* Cf. Landau (1), Satz 167.

LEMMA 4. *Let t be a positive integer, p an odd prime. Then the quadratic form $f(x_1,\dots,x_n)$ is equivalent to a form f' in new variables ξ of the type*

$$f' \equiv p^{\rho_1}\alpha_1\xi_1^2+p^{\rho_2}\alpha_2\xi_2^2+\dots+p^{\rho_n}\alpha_n\xi_n^2 \pmod{p^t}$$
$$(0 \leqslant \rho_1 \leqslant \rho_2 \leqslant \dots \leqslant \rho_n),$$

*the ρ's being integers, and the α's prime to p.**

LEMMA 5. *For sufficiently large t, we have $f \sim f'$, where f', a form in variables ξ, ξ', is of the type*

$$f' \equiv \sum_{s=1}^{\lambda} 2^{\rho_s}\phi_s \pmod{2^t} \qquad (\rho_s \geqslant 0),$$

where the ρ's are integers and ϕ_s is either of the type

$$\phi_s = \sum_{i=1}^{k_s} \alpha_{is}\xi_{is}\xi_{is} \quad (k_s \text{ an integer}),$$

or

$$\phi_s = \sum_{i=1}^{k_s/2} (2\alpha_{is}\xi_{is}\xi_{is}+2\alpha'_{is}\xi_{is}\xi'_{is}+2\alpha''_{is}\xi'_{is}\xi'_{is}) \quad (k_s \text{ even}),$$

the α_{is}, α'_{is}, α''_{is} being odd and $\sum_{s=1}^{\lambda} k_s = n$.†

Denote the minor determinant of the matrix (a_{ij}) $(i,j = 1,\dots,n)$ of $f(x_1,\dots,x_n)$ formed by the elements at the intersections of rows $i_1, i_2,\dots, i_k$ and columns $j_1, j_2,\dots, j_k$ by $\mathscr{A}^{(k)}_{i_1,\dots,i_k;j_1,\dots,j_k}$; the g.c.d. of all the minors of order k by d_k; the g.c.d. of all the integers

$$\mathscr{A}^{(k)}_{i_1,\dots,i_k;i_1,\dots,i_k}/d_k, \qquad 2\mathscr{A}^{(k)}_{i_1,\dots,i_k,j_1,\dots,j_k}/d_k,$$

by $\sigma_k = 1$ or 2 $(k = 1,\dots,n)$. Let $d_{-1} = 0$, $d_0 = 1$, $d_{n+1} = 0$, and define the numbers

$$o_k = d_{k+1}d_{k-1}/d_k^2 \quad (k = 0,\dots,n).$$

Then we have the following two known theorems:

LEMMA 6. *No two consecutive σ_k can be both equal to 2.*‡

LEMMA 7. *Let*§

$$\mathscr{A}^{(k)}_{1,2,\dots,k;1,2,\dots,k} = \sigma_k d_k B_k \quad (k = 1,\dots,n)$$

so that $B_n = 1$. Define $B_0 = 1$. Then there exists a form $f' \sim f$ with

$$(B'_k,\ 2B'_{k+1}B'_{k-1}o_1o_2\dots o_{n-1}) = 1 \quad (k = 1,\dots,n-1).$$

* Bachmann (1), vol. 4, part 1, 434. † Ibid. 444.

‡ Ibid. 445. It should be noted that the notations used here are different from those used by Bachmann. § Ibid. 452.

LEMMA 8. *Let*

$$f_5 = \sum_{i,j=1}^{5} a_{ij} x_i x_j \quad (a_{ij} = a_{ji})$$

be a quadratic form with determinant D_5 and let the odd prime p be a divisor of d_3. Then the transformation

$$x_i = y_i \quad (i = 1, 2, 3, 4), \qquad p x_5 = y_5$$

carries a form $f_5'' \sim f_5$ into a form with integer coefficients and determinant D_5/p^2.

By Lemma 4, with $t \geqslant 2$, we have $f_5' \sim f_5$ and

$$f_5' \equiv p^{\rho_1}\alpha_1\xi_1^2 + \ldots + p^{\rho_5}\alpha_5\xi_5^2 \pmod{p^t} \qquad (0 \leqslant \rho_1 \leqslant \ldots \leqslant \rho_5).$$

The lemma is evident if one of the ρ's is greater than 1. Suppose then that each of the ρ's is not greater than 1.

Since $p | d_3$, at least three of the ρ's must be 1, say

$$\rho_3 = \rho_4 = \rho_5 = 1.$$

Then the unitary substitution

$$\xi_1 = \eta_1, \qquad \xi_2 = \eta_2, \qquad \xi_3 = \eta_3 + z_3\eta_5, \qquad \xi_4 = \eta_4 + z_4\eta_5, \qquad \xi_5 = \eta_5$$

carries f_5' into f_5'' of the type

$$f_5'' \equiv p^{\rho_1}\alpha_1\eta_1^2 + p^{\rho_2}\alpha_2\eta_2^2 + p\alpha_3\eta_3^2 + p\alpha_4\eta_4^2 + 2p(\alpha_3 z_3\eta_3\eta_5 + \alpha_4 z_4\eta_4\eta_5) + \\ + p(\alpha_3 z_3^2 + \alpha_4 z_4^2 + \alpha_5)\eta_5^2 \pmod{p^t}.$$

By Lemma 3, since $(\alpha_3, p) = (\alpha_4, p) = 1$,

$$\alpha_3 z_3^2 + \alpha_4 z_4^2 + \alpha_5 \equiv 0 \pmod{p}$$

is solvable in z_3, z_4, and so the lemma is proved, since the coefficients of $\eta_i\eta_5$ $(i = 1, 2, 3, 4)$ are divisible by p and the coefficient of η_5^2 by p^2.

LEMMA 9. *Let 2 be a divisor of d_4. Then the transformation*

$$x_i = y_i \quad (i = 1, 2, 3, 4), \qquad 2x_5 = y_5$$

carries a form $f_5'' \sim f_5$ into a form with integer coefficients and determinant $D_5/4$.

By Lemma 5, with $t \geqslant 2$, we have $f_5' \sim f_5$, f_5' being one of the following three types:

$$f_5' \equiv 2^{\rho_1}\alpha_1\xi_1^2 + 2^{\rho_2}\alpha_2\xi_2^2 + 2^{\rho_3}\alpha_3\xi_3^2 + 2^{\rho_4}\alpha_4\xi_4^2 + 2^{\rho_5}\alpha_5\xi_5^2, \tag{5.1}$$

$$f_5' \equiv 2^{\rho_1}\alpha_1\xi_1^2 + 2^{\rho_2}\alpha_2\xi_2^2 + 2^{\rho_3}\alpha_3\xi_3^2 + 2^{\rho_4}(2\alpha_4\xi_4^2 + 2\alpha_5\xi_4\xi_5 + 2\alpha_6\xi_5^2), \tag{5.2}$$

$$f_5' \equiv 2^{\rho_1}\alpha_1\xi_1^2 + 2^{\rho_2}(2\alpha_2\xi_2^2 + 2\alpha_3\xi_2\xi_3 + 2\alpha_4\xi_3^2) + \\ + 2^{\rho_3}(2\alpha_5\xi_4^2 + 2\alpha_6\xi_4\xi_5 + 2\alpha_7\xi_5^2) \pmod{2^t}, \tag{5.3}$$

where the α's are odd.

Suppose first f_5' is of the type (5.1). The lemma is evident when

one of the ρ's is greater then 1. Suppose then $\rho_i \leqslant 1$ $(i = 1,..., 5)$. Since $2|d_4$, at least two of the ρ's must be 1, say $\rho_4 = \rho_5 = 1$. Then the unitary transformation

$$\xi_1 = \eta_1, \quad \xi_2 = \eta_2, \quad \xi_3 = \eta_3, \quad \xi_4 = \eta_4 + \eta_5, \quad \xi_5 = \eta_5$$

carries f_5' into

$$f_5'' \equiv 2^{\rho_1}\alpha_1\eta_1^2 + 2^{\rho_2}\alpha_2\eta_2^2 + 2^{\rho_3}\alpha_3\eta_3^2 + 2\alpha_4\eta_4^2 + 2.2\alpha_4\eta_4\eta_5 + \\ + 2(\alpha_4+\alpha_5)\eta_5^2 \pmod{2^t},$$

and so the lemma is proved since $\alpha_4 + \alpha_5 \equiv 0 \pmod 2$.

Suppose next that f_5' is of the type (5.2). The lemma is evident if either one of the ρ_i $(i = 1, 2, 3)$ is greater than 1 or $\rho_4 > 0$. Suppose then that $\rho_i \leqslant 1$ $(i = 1, 2, 3)$ and $\rho_4 = 0$. Since $2|d_4$, we must have two of the $\rho_i = 1$, say $\rho_1 = \rho_2 = 1$, for if, say $\rho_2 = \rho_3 = 0$, the cofactor of the coefficient of ξ_1^2 in the determinant of f_5' would be odd, i.e. $2 \nmid d_4$. Then as above, by a unitary substitution

$$\xi_1 = \eta_5, \quad \xi_2 = \eta_5 + \eta_2, \quad \xi_3 = \eta_3, \quad \xi_4 = \eta_4, \quad \xi_5 = \eta_1,$$

we can make all the coefficients of the terms involving η_5^2 divisible by 4, and so the lemma is proved.

Suppose finally, f_5' is of the type (5.3). The lemma is evident if either ρ_2 or ρ_3 is positive. Suppose now $\rho_2 = \rho_3 = 0$. Then the cofactor of the coefficient of ξ_1^2 in the determinant of f_5' would be odd, i.e. $2 \nmid d_4$.

Hence the proof of the lemma is complete.

LEMMA 10. *If $d_3 = 1$ and d_4 is odd, then a unitary substitution carries f_5 into a form*

$$f_5' = \sum_{i,j=1}^{5} a_{ij}' x_i x_j \qquad (a_{ij}' = a_{ji}'),$$

where

$$\begin{vmatrix} a_{22}' & a_{23}' & a_{24}' & a_{25}' \\ a_{23}' & a_{33}' & a_{34}' & a_{35}' \\ a_{24}' & a_{34}' & a_{44}' & a_{45}' \\ a_{25}' & a_{35}' & a_{45}' & a_{55}' \end{vmatrix} \not\equiv 0 \pmod 4$$

and is relatively prime to

$$\begin{vmatrix} a_{33}' & a_{34}' & a_{35}' \\ a_{34}' & a_{44}' & a_{45}' \\ a_{35}' & a_{45}' & a_{55}' \end{vmatrix}.$$

Since $d_3 = 1$, we have

$$o_1 = o_2 = 1, \qquad o_3 = d_4,$$

and so, by Lemma 7 with $k = 3$, there exists a form $f_5'' \sim f_5$ with

$$(\mathcal{A}^{(3)}_{1,2,3;1,2,3}/\sigma_3, \quad 2d_4\mathcal{A}^{(4)}_{1,2,3,4;1,3,n,4}/\sigma_4 d_4) = 1.$$

Suppose first $\sigma_3 = 1$, we have

$$(\mathcal{A}^{(3)}_{1,2,3;1,2,3},\quad \mathcal{A}^{(4)}_{1,2,3,4;1,2,3,4}) = 1. \tag{6.1}$$

Suppose next $\sigma_3 = 2$, we have, by Lemma 6, $\sigma_4 = 1$ and so

$$\mathcal{A}^{(4)}_{1,2,3,4;1,2,3,4}\ (= \sigma_4 d_4 B_4)$$

is odd, since d_4 is odd and B_4, by Lemma 7, is odd also. Then we have again

$$(\mathcal{A}^{(3)}_{1,2,3;1,2,3},\quad \mathcal{A}^{(4)}_{1,2,3,4;1,2,3,4}) = 1. \tag{6.2}$$

Hence (6.1), (6.2) show that, by a transformation

$$x_1 = y_5,\quad x_2 = y_4,\quad x_3 = y_3,\quad x_4 = y_2,\quad x_5 = y_1,$$

we obtain a form f'_5 satisfying the conditions of the lemma.

Denoting the cofactors of a_{ij} in $|a_{ij}|$ $(i,j = 1,\ldots,n)$ by $A^{(n)}_{ij}$, we have

LEMMA 11. *If* $A^{(5)}_{11} \not\equiv 0 \pmod 4$, *then for an arbitrary integer* u, *there exists an integer* v *such that*

$$D_5 u - A^{(5)}_{11} v^2 - 2A^{(5)}_{12} v - A^{(5)}_{22} \neq 4^\rho(8\tau-1). \tag{7}$$

It suffices to prove that, if a, b, c are integers, where $a \not\equiv 0 \pmod 4$, then integers x exist such that

$$Q = ax^2+bx+c \neq 4^\rho(8\tau-1). \tag{8}$$

Suppose first $c \neq 4^\rho(8\tau-1)$, it is clear that $x = 0$ is a solution of (8). Suppose next $c = 4^\rho(8\tau-1)$. We consider the following cases:

I. If $c = 8\tau-1$. Construct the table:

$x \equiv$ / $b \equiv$ / $Q \equiv$ / $a \equiv$	0	1	2	3	4	5	6	7
1	2 / 3	1 / 1	1 / 2	1 / 3	2 / 3	1 / 5	1 / 6	2 / 1
2	1 / 1	1 / 2	1 / 3	2 / 5	1 / 5	1 / 6	2 / 3	2 / 5
3	1 / 2	1 / 3		1 / 5	1 / 6	2 / 5		1 / 1
5	2 / 3	1 / 5	1 / 6	2 / 1	2 / 3	1 / 1	1 / 2	1 / 3
6	1 / 5	1 / 6	2 / 3	2 / 5	1 / 1	1 / 2	1 / 3	2 / 5
7	1 / 6	2 / 5		1 / 1	1 / 2	1 / 3		1 / 5

(mod 8).

For an illustration of the table, we take $a \equiv 1$, $b \equiv 0 \pmod 8$, then $Q \equiv 3$ for $x \equiv 2 \pmod 8$. Hence we need only consider the cases not dealt with in the table, namely $b \equiv \pm 2$ and $a \equiv 3$ or $7 \pmod 8$. We write

$$aQ = (ax+\tfrac{1}{2}b)^2+ac-(\tfrac{1}{2}b)^2.$$

Since $a \equiv 3$ or $7 \pmod 8$, it suffices to show that we can choose x such that $aQ \neq 4^\rho(8\tau+5)$ or $4^\rho(8\tau+1)$, or, since a is odd, that y can be chosen so that

$$y^2+d \neq 4^\rho(8\tau+5) \text{ or } 4^\rho(8\tau+1),$$

where $d = ac-(\frac{1}{2}b)^2$. Here we need only consider the cases of $d \not\equiv 0 \pmod 4$, since the factor 4 can be removed by putting $y = 2y_1$. If $d \equiv -1, \pm 2, 3 \pmod 8$, we take $y \equiv 2 \pmod 4$, so that $y^2+d \equiv 3, \mp 2, -1 \pmod 8$. If $d \equiv 1$ or $-3 \pmod 8$, we take $y \equiv 1 \pmod 4$, so that $y^2+d \equiv 2$ or $-2 \pmod 8$ respectively. Hence (8) is solvable when $c = 8\tau-1$.

II. Suppose $c = 4^{\rho_1}(8\tau_1-1)$, $\rho_1 > 0$ and b is odd. We take $x \equiv 2 \pmod 4$ and then $Q \equiv 2 \pmod 4 \neq 4^\rho(8\tau-1)$.

III. Suppose $c = 4^{\rho_1}(8\tau_1-1)$, $\rho_1 > 0$ and $b = 2^\beta b_1$, where $\beta > 0$, and b_1 is odd. If $\beta \geqslant \rho_1$, take $x = 2^{\rho_1}x_1$, which reduces it into the case I. If $\beta < \rho_1$, take $x = 2^\beta x_1$, which reduces it into the case II.

This proves the lemma.

LEMMA 12. *Given a positive-definite quadratic form*

$$f_5 = \sum_{i,j=1}^{5} a_{ij}x_ix_j \qquad (a_{ij} = a_{ji})$$

with

$$A_{11}^{(5)} = \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{23} & a_{33} & a_{34} & a_{35} \\ a_{24} & a_{34} & a_{44} & a_{45} \\ a_{25} & a_{35} & a_{45} & a_{55} \end{vmatrix} \not\equiv 0 \pmod 4$$

and relatively prime to

$$\begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{34} & a_{44} & a_{45} \\ a_{35} & a_{45} & a_{55} \end{vmatrix},$$

then we can find two integers a_{16}, a_{66} such that the form

$$f_6 = f_5+2a_{16}x_1x_6+2x_2x_6+a_{66}x_6^2.$$

(i) *is positive-definite,*

(ii) *has determinant $D_6 \neq 4^\rho(8\tau-1)$,*

and (iii) $(A_{11}^{(6)}, D_6) = 1$, *where*

$$A_{11}^{(6)} = \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} & 1 \\ a_{23} & a_{33} & a_{34} & a_{35} & 0 \\ a_{24} & a_{34} & a_{44} & a_{45} & 0 \\ a_{25} & a_{35} & a_{45} & a_{55} & 0 \\ 1 & 0 & 0 & 0 & a_{66} \end{vmatrix} = a_{66} A_{11}^{(5)} - \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{34} & a_{44} & a_{45} \\ a_{35} & a_{45} & a_{55} \end{vmatrix}$$

$$= p, \text{ an odd prime.}$$

Denote

$$\begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{34} & a_{44} & a_{45} \\ a_{35} & a_{45} & a_{55} \end{vmatrix} \quad \text{by } \Delta.$$

Suppose first that Δ is odd. From Lemma 11 there exist integers w_0 and v_0 such that

$$2D_5 w_0 - A_{11}^{(5)} v_0^2 - 2A_{12}^{(5)} v_0 - A_{22}^{(5)} \not\equiv 4^\rho (8\tau - 1).$$

Then, on taking a fixed v_0 with $A_{11}^{(5)} v_0 + A_{12}^{(5)} \not\equiv 0$, we can choose a suitable integer γ such that any $w_1 \equiv w_0 \pmod{2^\gamma}$ is also a solution. By hypothesis, $(A_{11}^{(5)}, \Delta) = 1$, and thus we can choose w_1 so that

$$2A_{11}^{(5)} w_1 - \Delta = p > \Delta |A_{11}^{(5)} v_0 + A_{12}^{(5)}| > 0,$$

where p is a prime. Now

$$2A_{12}^{(5)} w_1 + \Delta v_0 \not\equiv 0 \pmod{p},$$

for, if it were, p would divide

$$|-A_{12}^{(5)}(2A_{11}^{(5)} w_1 - \Delta) + A_{11}^{(5)}(2A_{12}^{(5)} w_1 + \Delta v_0)| = \Delta |A_{12}^{(5)} + A_{11}^{(5)} v_0| < p.$$

We show that it suffices to take $a_{16} = v_0$, $a_{66} = 2w_1$. For from

$$\begin{vmatrix} A_{11}^{(6)} & A_{12}^{(6)} \\ A_{12}^{(6)} & A_{22}^{(6)} \end{vmatrix} \cdot D_6$$

$$= \begin{vmatrix} A_{11}^{(6)} & A_{12}^{(6)} & A_{13}^{(6)} & A_{14}^{(6)} & A_{15}^{(6)} & A_{16}^{(6)} \\ A_{12}^{(6)} & A_{22}^{(6)} & A_{23}^{(6)} & A_{24}^{(6)} & A_{25}^{(6)} & A_{26}^{(6)} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & 1 \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & 0 \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & 0 \\ a_{16} & 1 & 0 & 0 & 0 & a_{66} \end{vmatrix}$$

$$= \begin{vmatrix} D_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_6 & 0 & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & 0 \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & 0 \\ a_{16} & 1 & 0 & 0 & 0 & a_{66} \end{vmatrix} = a_{66} \Delta D_6^2,$$

$$\begin{vmatrix} A_{11}^{(6)} & A_{12}^{(6)} \\ A_{12}^{(6)} & A_{22}^{(6)} \end{vmatrix} = \Delta a_{66} D_6,$$

and so $$\Delta a_{66} D_6 \equiv -(A_{12}^{(6)})^2 \pmod{A_{11}^{(6)}}.$$

But

$$A_{12}^{(6)} = -\begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{23} & a_{33} & a_{34} & a_{35} & 0 \\ a_{24} & a_{34} & a_{44} & a_{45} & 0 \\ a_{25} & a_{35} & a_{45} & a_{55} & 0 \\ 1 & 0 & 0 & 0 & a_{66} \end{vmatrix} = a_{66} A_{12}^{(5)} + a_{16} \Delta.$$

From the values of a_{16}, a_{66}, we have

$$(A_{11}^{(6)}, A_{12}^{(6)}) = (2A_{11}^{(5)} w_1 - \Delta, \quad 2A_{12}^{(5)} w_1 + \Delta v_0) = 1.$$

Hence $(A_{11}^{(6)}, \Delta a_{66} D_6) = 1$, i.e. $(A_{11}^{(6)}, D_6) = 1$ and (ii), (iii) are satisfied.

Since f_5 is positive-definite, as in Lemma 1, after v_0 is fixed, if we choose positive w_1 so large that $D_6 > 0$, the condition (i) is satisfied.

Suppose next that Δ is even. Write $a_{66} = 2w+1$, and proceeding as above, using Lemma 11 with $u = 2w+1$ and making $A_{11}^{(5)}(2w+1) - \Delta$ a prime, we have the same result.

LEMMA 13. *If, in Lemma* 12, $D_6 \equiv 0 \pmod 4$, *the transformation*

$$x_i = y_i \quad (i = 1, 2, 3, 4), \qquad 2x_i = y_i \tag{9}$$

carries a form $f_6'' \sim f_6$ *into a form with integer coefficients, determinant* $D_6/4$ *and* $d_5 = 1$, $\sigma_5 = 1$.

The proof of the first part of the lemma is similar to that in Lemma 9.

For the second part of the lemma, $\sigma_5 = 1$ is obvious, since, by Lemma 12, $A_{11}^{(6)} = p$ is odd. If $d_5 > 1$, we have $d_5 = p$ and hence $D_6 \equiv 0 \pmod p$ contrary to $(D_6, A_{11}^{(6)}) = 1$.

LEMMA 14. *Let*

$$f_6 = \sum_{i,j=1}^{6} a_{ij} x_i x_j \quad (a_{ij} = a_{ji})$$

be a positive-definite form with $(A_{11}^{(6)}, D_6) = 1$, *where* $D_6 \not\equiv 0 \pmod 4$ *is the determinant of* f_6 *and* $A_{11}^{(6)}$, *the cofactor of* a_{11} *in* D_6, *is odd. If* $D_6 \not\equiv 7 \pmod 8$, *there exist integers* a_{77}, a_{78}, a_{88} *such that the form in eight variables*

$$g_8 = f_6 + 2x_1 x_8 + a_{77} x_7^2 + 2a_{78} x_7 x_8 + a_{88} x_8^2$$

is a positive-definite properly-primitive form with determinant unity.

Since the determinant of g_8 is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 & 1 \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} & 0 & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} & 0 & 0 \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} & 0 & 0 \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & a_{78} \\ 1 & 0 & 0 & 0 & 0 & 0 & a_{78} & a_{88} \end{vmatrix} = 1,$$

the proof is precisely the same as that in Lemma 1 simply by replacing a_{22}, D_2, g_4, a_{33}, a_{34}, a_{44} in Lemma 1 by $A_{11}^{(6)}$, D_6, g_8, a_{77}, a_{78}, a_{88} respectively, except that we need to show that one of the a_{77}, a_{78} can be odd, i.e. to show that in Lemma 1 we can make a_{33} or a_{44} odd, when a_{22} is odd.

In the proof of Lemma 1, in case $D_2 \equiv 3 \pmod 8$, $a_{44} D_2 - a_{22} = 2p$, we have a_{44} odd, since a_{22} is odd. In all the other cases, $a_{44} D_2 - a_{22} = p$. If we put $a_{34} = 2a'_{34}$, the solvability of (4) is not affected, since $(4/p) = 1$ and then a_{33} must be odd, since the left-hand side of (4) is odd. Hence, correspondingly, one of a_{77}, a_{88} can be chosen to be odd in this lemma.

THEOREM 2. $R_5 = 8$.

$R_5 \geqslant 8$ is clear from the form $x_1^2+x_2^2+x_3^2+x_4^2+7x_5^2$, since 7 is not a sum of less than three integer squares.

Next we prove that $R_5 \leqslant 8$. From Lemmas 8, 9, and 10, it is clear that we need only to prove the theorem for the form

$$f_5 = \sum_{i,j=1}^{5} a_{ij} x_i x_j \quad (a_{ij} = a_{ji}),$$

where $A_{11}^{(5)}$ is relatively prime with

$$\begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{34} & a_{44} & a_{45} \\ a_{35} & a_{45} & a_{55} \end{vmatrix}.$$

Since from Lemma 2, there is only one properly-primitive positive-definite quadratic form in eight variables with determinant unity, then by Lemmas 12 and 14, if $D_6 \not\equiv 0 \pmod 4$ in Lemma 12,

$$g_8 = f_5+2x_1x_8+2x_2x_6+2a_{16}x_1x_6+2a_{78}x_7x_8+a_{66}x_6^2+a_{77}x_7^2+a_{88}x_8^2$$

is a positive-definite properly-primitive form with determinant unity

and so

$$g_8 = \sum_{i=1}^{8} L_i^2,$$

where the L's are linear forms in $x_1, \ldots, x_8$. Putting $x_6 = x_7 = x_8 = 0$, we have a representation of f_5 as a sum of eight squares of linear forms.

If, in Lemma 12, $D_6 \equiv 0 \pmod 4$, then by Lemma 13, we can carry f_6 in the Lemma 12, by some unitary substitutions combined with those of the type (9), into a form f_6' with $D_6' \not\equiv 0 \pmod 4$ and $d_5' = \sigma_5' = 1$. Then by similar argument to that in Lemma 10, we can assume that f_6' satisfies all the conditions of Lemma 14. Then there exist integers a_{77}, a_{78}, a_{88} such that

$$f_6' + 2x_1x_8 + a_{77}x_7^2 + 2a_{78}x_7x_8 + a_{88}x_8^2 = \sum_{i=1}^{8} L_i'^2.$$

Putting $x_7 = x_8 = 0$ and transforming f_6' back to f_6, we have

$$f_6 = \sum_{i=1}^{8} L_i''^2. \tag{10}$$

Putting $x_6 = 0$ in (10), we have a representation of f_5 as a sum of eight squares of linear forms. Hence $R_5 = 8$.

THEOREM 3. $R_4 = 7$, $R_3 = 6$, $R_2 = 5$.

The proof for $R_4 \leqslant 7$ is evident by applying Theorem 2 to the forms of the type $f_4 + x_5^2$, where

$$f_4 = \sum_{i,j=1}^{4} a_{ij}x_ix_j \quad (a_{ij} = a_{ji}).$$

Again, $R_4 \geqslant 7$ follows from the form $x_1^2 + x_2^2 + x_3^2 + 7x_4^2$.

Similarly, we can prove $R_3 = 6$, $R_2 = 5$.

It should be noticed that the argument of this section with some modification also proves

THEOREM 4.1. *Any positive-definite quadratic form in $n = 3, 4,$ or 5 variables with determinant $D_n \not\equiv 0, 4, 7 \pmod 8$, $(A_{11}^{(n)}, D_n) = 1$, can be expressed as a sum of $n+2$ linear squares with integer coefficients.*

THEOREM 4.2. *Any positive-definite quadratic form in six variables with determinant $D_6 \not\equiv 0, 4, 7 \pmod 8$, $(A_{11}^{(6)}, D_6) = 1$, and $A_{11}^{(6)}$ odd when $(A_{11}^{(6)}/D_6) = 1$, $D_6 \equiv 3 \pmod 8$ can be expressed as a sum of eight linear squares with integer coefficients.*

6. The conjecture $R_n = n+3$ is not true when $n > 5$, for Mordell has given recently the form

$$\sum_{i=1}^{6} x_i^2 + \Big(\sum_{i=1}^{6} x_i\Big)^2 - 2x_1x_2 - 2x_2x_6,$$

which he proved cannot be decomposed into a sum of any number of squares of linear forms with integer coefficients.* Before this I had also proved the following:

THEOREM 5. *There exists at least one positive-definite quadratic form in n variables where $6 \leqslant n \leqslant 8$, which cannot be represented as a sum of non-negative forms with integer coefficients (i.e. with integer a_{ij}).*

The proof requires the known lemma

LEMMA 15. *Any positive semi-definite quadratic form in n variables with integer coefficients can be derived by a linear transformation with integer coefficients from a definite quadratic form in fewer than n variables.*

Since the form f_n in n variables is semi-definite, it represents zero and also its determinant is zero. Then by a linear transformation of determinant unity and integer coefficients, we can write

$$f_n \sim 2\sum_{i=2}^{n} a_{1i}x_1x_i + f_{n-1}(x_2,\ldots,x_n).$$

Hence $a_{1i} = 0$, as otherwise the form assumes negative values by choice of x_1. Then f_{n-1} is definite or semi-definite in one variable less. The lemma is evident by continuing this process.

Now we can prove Theorem 5. Suppose the theorem false; then every positive-definite form in n ($\leqslant 8$) variables can be split into a sum of non-negative forms. Take a form $f_8^{(1)}$ in 8 variables with determinant unity and integer coefficients. Write

$$f_r^{(s)} = \sum_{i,j=1}^{r} a_{ij}^{(s)} x_i x_j \qquad (a_{ij}^{(s)} = a_{ji}^{(s)}).$$

Then we have

$$f_8^{(1)} = f_8^{(2)} + f_8^{(3)},$$

where $a_{ii}^{(1)} = a_{ii}^{(2)} + a_{ii}^{(3)}$. If $f_8^{(2)}$ is positive-definite, we carry on the process again. If $f_8^{(2)}$ is semi-definite, by Lemma 15, we have

$$f_8^{(2)} \sim f_t^{(2)},$$

where $t \leqslant 7$ and $f_t^{(2)}$ is positive-definite. Then we apply our process to $f_t^{(2)}$. It is clear that finally, we must have

$f_8^{(1)} = L^2 +$ *a non-negative quadratic form with integer coefficients,*

where L is a linear form in $x_1,\ldots,x_8$ with integer coefficients having no common factor. By a linear transformation we can write $L = x_1$, so that

$$f_8^{(1)} \sim f_8,$$

* The proof of this will appear in the *Annals of Math.*

and $f_8 - x_1^2$ is a non-negative form. Let the determinant of f_8 be $|a_{ij}|$ $(i,j = 1,\ldots,8)$, then the determinant of $f_8 - x_1^2$ is

$$\begin{vmatrix} a_{11}-1 & a_{12} & . & . & a_{18} \\ a_{12} & a_{22} & . & . & a_{28} \\ . & . & . & . & . \\ a_{18} & a_{28} & . & . & a_{88} \end{vmatrix} \geqslant 0,$$

and so

$$1 \geqslant A_{11}^{(8)} > 0,$$

i.e.

$$A_{11}^{(8)} = 1.$$

Since all the forms in seven variables with determinant unity $\sim \sum_{i=1}^{7} x_i^2$, then $f_8 \sim \sum_{i=1}^{8} x_i^2$, since it represents 1, contrary to the fact that the class number of a positive-definite form with determinant unity in eight variables exceeds 1. Hence Theorem 5 is proved.

In closing, I wish to express my sincere gratitude to Professor Mordell for suggesting to me this problem and the method of attack, and for helping me with my manuscript.

A THEOREM ON THE ADDITION OF RESIDUE CLASSES: APPLICATION TO THE NUMBER $\Gamma(k)$ IN WARING'S PROBLEM

By I. CHOWLA (*Lahore*)

[Received 13 July 1936; in revised form 7 December 1936]

1. Introduction. THE following remarkable theorem* has recently been proved by Davenport:

Let p be a prime; let $\alpha_1,\ldots,\alpha_u$ be u different residue classes to modulus p; let $\beta_1,\ldots,\beta_v$ be v different residue classes to modulus p. Let $\gamma_1,\ldots,\gamma_w$ be all those different residue classes which are representable as

$$\alpha_i+\beta_j \quad (1 \leqslant i \leqslant u;\ 1 \leqslant j \leqslant v).$$

Then $w \geqslant u+v-1$ provided that $u+v-1 \leqslant p$; and otherwise $w = p$.

In this paper I generalize† this theorem to any composite modulus. To do this I must introduce certain restrictions on one of the sequences. My theorem is as follows:

THEOREM 1. *Let m be a positive integer; let $\alpha_1,\ldots,\alpha_u$ be u different residue classes to modulus m; let $\beta_1,\ldots,\beta_v$ be v different residue classes to modulus m, where one of the β's is zero and the rest are prime to m. Let $\gamma_1,\ldots,\gamma_w$ be all those different residue classes which are representable as*

$$\alpha_i+\beta_j \quad (1 \leqslant i \leqslant u;\ 1 \leqslant j \leqslant v).$$

Then $w \geqslant u+v-1$ if $u+v-1 \leqslant m$; and otherwise $w = m$.

Theorem 1 has an application to the number $\Gamma(k)$ in Waring's problem. $\Gamma(k)$ was defined by Hardy and Littlewood as the least s such that every arithmetical progression contains an infinity of numbers of the form $x_1^k+\ldots+x_s^k$ where the x's are integers. Hardy and Littlewood‡ have proved that $\Gamma(k) \leqslant 4k$, while $\Gamma(k) = 4k$ for infinitely many k. The following theorem is an easy consequence of Theorem 1.

THEOREM 2. *Let $s \geqslant 4k$ and let the integers $a_1,\ldots,a_s$ be prime to m. Then every arithmetical progression $mx+a$ contains infinitely many numbers of the form $a_1x_1^k+\ldots+a_sx_s^k$ where the x's are integers.*

* *J. of London Math. Soc.* 10 (1935), 30–2.

† Compare *Proc. Indian Acad. Sc.* (A), 2 (1935), 242–3, where it was shown by examples that the generalization (Theorem 1 of this paper) is, in a sense, best possible.

‡ *Math. Zeits.* 12 (1922), 161–88 (Theorem 12, 186).

The particular case $a_1 = \ldots = a_s = 1$ gives $\Gamma(k) \leqslant 4k$.

It also follows from our argument that

THEOREM 3. *Every arithmetical progression $mx+a$ where m is odd and prime to $a_1, a_2, \ldots, a_s$, contains an infinity of numbers of the form $a_1x_1^k+\ldots+a_sx_s^k$ provided that $s \geqslant \frac{3}{2}k$.*

2. Proof of Theorem 1

We note that the last clause of the enunciation is an immediate consequence of the preceding one, for, if $u+v-1 > m$, we need consider only the two sets $\alpha_1, \ldots, \alpha_u$; $\beta_1, \ldots, \beta_{m+1-u}$.

The proof is by induction on v. For $v = 1$ there is nothing to prove. For the rest of the proof we can assume without loss of generality that $\beta_v = 0$. For $v = 2$ we write $\beta = \beta_1 - \beta_2 = \beta_1 \neq 0$, and observe that, if the theorem is false, then, for every α_i, $\alpha_i+\beta$ is also an α. Hence for every integer f, $\alpha_i+f\beta$ is an α. Since β is prime to m every residue class to modulus m is representable by $\alpha_i+f\beta$, so that $u = m$; which contradicts the hypothesis that $u+2-1 \leqslant m$.

Hence we may suppose that $v > 2$ and that the theorem is true for all $v' < v$. We may also suppose that $w < m$. We apply the theorem to the two sets of residue classes $\gamma_1, \ldots, \gamma_w$; β_1, β_v. Since $w+2-1 \leqslant m$, we conclude that there are at least $w+1$ residue classes which are either of the form $\gamma_i+\beta_1$ or of the form

$$\gamma_i+\beta_v \quad (\beta_v = 0),$$

and therefore that there is a class δ such that $\delta-\beta_1$ is a γ and $\delta-\beta_v = \delta$ is not. We may suppose, without loss of generality (since we can arrange $\beta_2, \ldots, \beta_{v-1}$, and also $\gamma_1, \ldots, \gamma_w$ in any order we please), that there exists a suffix r $(1 \leqslant r < v)$ such that

$$\begin{aligned} \delta-\beta_g &= \gamma_g \quad (1 \leqslant g \leqslant r), \\ \delta-\beta_t &= \epsilon_t \quad (r < t \leqslant v), \\ \epsilon_t &\neq \gamma_y \quad (r < t \leqslant v;\ 1 \leqslant y \leqslant w). \end{aligned}$$

We now observe that none of the residue classes

$$\gamma_g-\beta_i \quad (r < t \leqslant v;\ 1 \leqslant g \leqslant r)$$

is an α. For, if so, we should have

$$\alpha+\beta_t = \gamma_g = \delta-\beta_g,$$

i.e.

$$\alpha+\beta_g = \delta-\beta_t = \epsilon_t,$$

and so ϵ_t would be a γ, which is not the case. Hence the w' residue classes representable in the form $\alpha_i+\beta_t$ $(1 \leqslant i \leqslant u;\ r < t \leqslant v)$ form

a sub-set of the γ's not containing $\gamma_1,\ldots,\gamma_r$. Thus $w' \leqslant w-r$. But by Theorem 1, with $v' = v-r$, we have

$$w' \geqslant u+(v-r)-1.$$

Hence
$$w \geqslant u+v-1.$$

3. Application to $\Gamma(k)$

LEMMA 1. *Let p be an odd prime, $\theta \geqslant 1$. Then the number of k-th-power residues, prime to p^θ, and incongruent to modulus p^θ is not less than h/k where $h = \phi(p^\theta)$ and ϕ is Euler's totient function.*

Proof. Let G be a primitive root to modulus p^θ. Then the numbers

$$G,\ G^2,\ldots,\ G^h \tag{A}$$

are mutually incongruent. Hence the numbers*

$$G^k,\ G^{2k},\ldots,\ G^{[h/k]k} \tag{B}$$

are also mutually incongruent since (B) is a subset of (A). But the numbers (B) are residues (prime to p^θ) of kth powers. We consider the two cases:

(i) $k|h$: then the number of numbers (B) is h/k;

(ii) $k\nmid h$: then it is easy to see that

$$G^{([h/k]+1)k} \not\equiv G^{bk} \quad (b = 1, 2,\ldots,[h/k]),$$

for we have $\quad 0 < \left(\left[\frac{h}{k}\right]+1-b\right)k \leqslant \left[\frac{h}{k}\right]k < \frac{h}{k}k = h.$

(since $k\nmid h$). Hence there are at least $[h/k]+1 \geqslant h/k$ distinct residues (prime to p^θ) of kth powers.

LEMMA 2. *The number of k-th-power residues to modulus 2^θ which are prime to 2^θ is at least equal to*

$$h/2k \quad \textit{where } h = \phi(2^\theta) = 2^{\theta-1}.$$

Proof. If θ is 0 or 1 there is at least one kth-power residue, namely 1, and

$$1 \geqslant \frac{2}{2k} \geqslant \frac{h}{2k}.$$

Hence we may suppose $\theta \geqslant 3$. It is known† that 5 belongs to the exponent $2^{\theta-2}$ to modulus 2^θ; hence the numbers

$$5,\ 5^2,\ldots,\ 5^{\frac{1}{2}h}$$

* We use the symbol $[z]$ to denote the greatest integer not exceeding z.

† Landau, *Vorlesungen über Zahlentheorie*, 1, Satz 125.

are mutually incongruent to modulus 2^θ. From this point onwards the proof is the same as that of Lemma 1, but with $\frac{1}{2}h$ instead of h.

Now consider the numbers x_1^k where x_1 takes the values 0 and other values, prime to p, not exceeding p^θ ($p > 2$). The number of incongruent numbers x_1^k is at least $1+h/k$ where $h = \phi(p^\theta)$, by Lemma 1. Hence the number of incongruent numbers $a_1 x_1^k$ is at least $1+h/k$ provided a_1 is prime to p. The same is true of the incongruent numbers $a_2 x_2^k, \ldots, a_s x_s^k$ provided $a_2, \ldots, a_s$ are also prime to p. Then by repeated application of Theorem 1, the set $a_1 x_1^k + \ldots + a_s x_s^k$ (where $x_2, \ldots, x_s$ run through the same set of values as x_1) contains at least

$$\min\left[p^\theta, s\left(\frac{h}{k}+1\right)-s+1\right], \quad \text{i.e.} \quad \min\left[p^\theta, \frac{sh}{k}+1\right]$$

incongruent numbers to modulus p^θ.

Hence, if s is such that

$$s\frac{h}{k}+1 \geqslant p^\theta,$$

i.e.

$$s \geqslant \frac{p^\theta-1}{h/k},$$

i.e. if

$$s \geqslant \frac{p^\theta}{h/k} = \frac{k}{1-1/p},$$

then every residue class to modulus p^θ is representable as

$$a_1 x_1^k + \ldots + a_s x_s^k.$$

When $p = 2$, by reasoning as above, Lemma 2 shows that, if

$$s \geqslant \frac{2k}{1-\frac{1}{2}} = 4k,$$

then every residue class to modulus 2^θ is representable as

$$a_1 x_1^k + \ldots + a_s x_s^k$$

Theorems 2 and 3 are immediate consequences.

A NOTE ON INTEGRAL FUNCTIONS OF ORDER ONE

By V. GANAPATHY IYER (*Madras*)

[Received 7 September 1936]

1. *Notation.* LET $f(z)$ be an integral function of order one. Let $z = re^{i\theta}$ and $M(r,f) = \max_{|z|\leqslant r} |f(z)|$. The type $k(f)$ and the function $h(\theta,f)$ are defined by the relations

$$k(f) = \overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r}, \qquad h(\theta,f) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r}.$$

1.1. The object of this note is to prove the following two theorems.

THEOREM 1. *Let* $\{\lambda_n\}$ $(0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty)$ *be a measurable* sequence of density* D *and index of condensation** δ $(0 \leqslant \delta < \infty)$. *Let*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log|f(\lambda_n)| = -\alpha < 0. \tag{1}$$

Then $k(f) \geqslant d = \min(\pi D, \alpha - 2\delta)$ *or else* $f(z)$ *is identically zero.*

THEOREM 2. *Let* $\{\lambda_n\}$ *be as in Theorem* 1. *Let*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log|f(\pm\lambda_n)| = -\alpha < 0, \tag{2}$$

where $\alpha > \delta$. *Then* $k(f) \geqslant \pi D$ *or else* $f(z) \equiv 0$.

1.2. These two theorems were suggested by the following result due to Miss Cartwright.†

THEOREM A. *Let* $f(z)$ *be an integral function of order one and type less than* π. *Let* $f(\pm n) = O(1)$ $(n = 1, 2, \ldots)$. *Then* $f(z)$ *is bounded along the whole of the real axis.*

1.3. All these theorems show that the type of $f(z)$ is governed by the behaviour of the function at an isolated sequence of points like $\{\lambda_n\}$ or $\{n\}$. It will be seen that a hypothesis of the form (1) or (2) is more directly related to the type of $f(z)$ than the assumption of mere boundedness.

* For the definition of measurability and the index of condensation of a sequence $\{\lambda_n\}$, see V. Bernstein, *Séries de Dirichlet*, Borel Tracts (1933), 22–7. The index is always zero when $\underline{\lim}(\lambda_{n+1} - \lambda_n) > 0$.

† *Quart. J. of Math.* (Oxford), 7 (1936), 46–55.

1.4. We require the following known results.*

LEMMA 1. *When $f(z)$ is of order one and $k(f)$ is finite, $h(\theta,f)$ and $h(\theta+\pi,f)$ cannot both be negative unless $f(z) \equiv 0$.*

LEMMA 2. *Let $\{\lambda_n\}$ be as in Theorem* 1. *Let*

$$\sigma(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right). \tag{3}$$

Then,

$$h(\theta,\sigma) = \pi D|\sin\theta| = \lim_{r\to\infty}\frac{\log|\sigma(re^{i\theta})|}{r} \tag{4}$$

where θ is not a multiple of π. Also

$$\delta = \overline{\lim_{n\to\infty}}\frac{1}{\lambda_n}\log\frac{1}{|\sigma'(\lambda_n)|}. \tag{5}$$

Moreover, given a positive ϵ, we can find $\gamma = \gamma(\epsilon)$, $r_0 = r_0(\epsilon)$ such that

$$|\sigma(z)| \leqslant e^{\epsilon|z|}, \qquad |\mathrm{amp}(z)| \leqslant \gamma, \qquad r = |z| \geqslant r_0. \tag{6}$$

2. *Proof of Theorem* 1. If $d \leqslant 0$ the result to be proved is trivial, since $k(f) \geqslant 0$ unless $f(z) \equiv 0$, in which case $k(f) = -\infty$. Let $d > 0$ and suppose $k(f) < d$. We shall prove that $f(z) \equiv 0$. Let $F(z) = e^{\beta z}f(z)$ where β is positive and is so chosen that

$$k(f)+\delta < \beta < d+\delta.$$

In virtue of the hypothesis and the relation (5), we have

$$\left.\begin{aligned}\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n}\log\left|\frac{F(\lambda_n)}{\sigma'(\lambda_n)}\right| &\leqslant \beta-\alpha+\delta < d-(\alpha-2\delta) \leqslant 0\\ \overline{\lim_{n\to\infty}}\frac{1}{\lambda_n}\log\left|\frac{F(-\lambda_n)}{\sigma'(-\lambda_n)}\right| &\leqslant -\beta+k(f)+\delta < 0\end{aligned}\right\}. \tag{7}$$

Remembering that $|e^{\beta z}| = e^{\beta r\cos\theta}$ and using (4), (7) and the method I have employed elsewhere,† it follows that

$$\frac{F(z)}{\sigma(z)} = \sum_{n=1}^{\infty}\left\{\frac{F(\lambda_n)}{\sigma'(\lambda_n)}\frac{1}{z-\lambda_n}+\frac{F(-\lambda_n)}{\sigma'(-\lambda_n)}\frac{1}{z+\lambda_n}\right\}. \tag{8}$$

Let $h > 1$ so that $\sum \lambda_n^{-h}$ converges. Let $C[\{\lambda_n\},h]$ denote the system of circles with centres $z = \pm\lambda_n$ and radii λ_n^{-h}. By (7) and (8) we conclude that, when z is outside the system $C[\{\lambda_n\},h]$,

$$|F(z)| \leqslant H|\sigma(z)|, \tag{9}$$

* See Bernstein, loc. cit. 301, 270–80, 289.

† *J. Indian Math. Soc.* (New Series), 2 (1936), 4–7.

where H is a constant independent of z. Using (6) we get that

$$|F(z)| \leqslant He^{\epsilon|z|}, \quad |\text{amp}(z)| \leqslant \gamma, \quad |z| \geqslant r_0, \tag{10}$$

z being outside the system $C[\{\lambda_n\}, h]$. Now choose $\gamma_1 \leqslant \gamma$ so that $1-2\cos\gamma_1 < 0$. Then (10) shows that there is a constant $A = A(\epsilon)$ such that

$$|e^{-2\epsilon z}F(z)| \leqslant A \tag{11}$$

for z in $|\text{amp}(z)| \leqslant \gamma_1$ but outside $C[\{\lambda_n\}, h]$. But $e^{-2\epsilon z}F(z)$ is regular in $|\text{amp}(z)| \leqslant \gamma_1$. Hence (11) holds throughout the angle $|\text{amp}(z)| \leqslant \gamma_1$. Therefore

$$h(0, F) \leqslant 0,$$

since ϵ is arbitrarily small. This gives

$$h(0,f) \leqslant -\beta.$$

But β is any number less than $d+\delta$. Therefore

$$h(0,f) \leqslant -(d+\delta). \tag{12}$$

Now, choose a positive η so that $k(f) < \eta < d$. Consider the function $e^{\eta z}f(z) = g(z)$. We have, by (12),

$$h(0,g) \leqslant \eta-(d+\delta) < 0$$

and

$$h(\pi,g) \leqslant -\eta+k(f) < 0.$$

Hence $h(0,g)$ and $h(\pi,g)$ are both negative, so that Lemma 1 gives $g(z) \equiv 0$. Therefore $f(z) \equiv 0$. This proves the theorem.

2.1. *Proof of Theorem* 2. Let $k(f) < \pi D$. Let β be positive and such that $\alpha-\delta > \beta$. Consider the function $e^{\beta z}f(z)$. We have, by hypothesis,

$$\left.\begin{aligned} \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n}\log\left|\frac{e^{\beta\lambda_n}f(\lambda_n)}{\sigma'(\lambda_n)}\right| &\leqslant \beta+\delta-\alpha < 0 \\ \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n}\log\left|\frac{e^{-\beta\lambda_n}f(-\lambda_n)}{\sigma'(-\lambda_n)}\right| &\leqslant -\beta+\delta-\alpha < 0 \end{aligned}\right\}. \tag{13}$$

Using (13) we can show, as in § 2, that $h(0,f) \leqslant -\beta$ and therefore $h(0,f) \leqslant -(\alpha-\delta)$.

Similarly, considering $e^{+\beta z}f(-z)$ we conclude that $h(\pi,f) \leqslant -(\alpha-\delta)$. We can now apply Lemma 1 and conclude that $f(z) \equiv 0$.

3. The following are particular cases of Theorems 1 and 2:

THEOREM 1A. *Let $f(z)$ be a function of order one. Let $\{\lambda_n\}$ be a measurable sequence of density D. Let $\varliminf_{n\to\infty}(\lambda_{n+1}-\lambda_n) > 0$ or, more generally, let the index of condensation be zero. Let*

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n}\log|f(\lambda_n)| = -\alpha < 0.$$

Then $k(f) \geqslant \min(\alpha, \pi D)$ or else $f(z) \equiv 0$. In particular, this is true with $\lambda_n = n$, $D = 1$.

THEOREM 2A. *Let $\{\lambda_n\}$ be as in Theorem 1A. Let $f(z)$ be a function of order one. Let*

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n}\log|f(\pm\lambda_n)| < 0.$$

Then $k(f) \geqslant \pi D$ or else $f(z) \equiv 0$. In particular, this is true with $\lambda_n = n$, $D = 1$.

3.1. *Remark.* Theorems 1A and 2A could be derived by combining Lemma 1 with a general result* proved by Bernstein. But the proof of the latter result (holding only when $\delta = 0$) is rather complicated. The proof given above for all non-negative δ is decidedly simpler. The lower limit $\min(\alpha-2\delta, \pi D)$ seems to depend on the exigencies of the method of proof. Probably, $k(f) \geqslant \min(\alpha-\delta, \pi D)$ is true under the conditions of Theorem 1. When $\delta = 0$ the difference in question is not evident. In the latter case, it is seen by considering $e^{-\alpha z}$ $(\alpha > 0)$ that $k(f) \geqslant \min(\alpha, \pi D)$ is the best possible result.

* See Bernstein, loc. cit. 235 (theorem 2) and 244 (theorem 4).

THE MEAN-VALUE OF $|\zeta(\frac{1}{2}+it)|^4$

By E. C. TITCHMARSH (*Oxford*)

[Received 26 July 1936]

1. THE formula

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \sim \frac{1}{2\pi^2}T\log^4 T \tag{1.1}$$

was proved by Ingham,* as a refinement on the corresponding O-result obtained by Hardy and Littlewood. Later I gave an alternative proof,† depending on Fourier transforms.

In the present paper I give another variant of the transform method; but for the sake of simplicity I go only as far as the O-result. Actually I give a new proof that

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt = O\left(\frac{1}{\delta}\log^4\frac{1}{\delta}\right). \tag{1.2}$$

The basic formula is

$$\begin{aligned}&\tfrac{1}{4}\pi\int_{-\infty}^{\infty}\frac{|\zeta(\frac{1}{2}+it)|^4 e^{(2\pi-4\delta)t}}{\cosh^2\pi t}\,dt\\ &=\int_1^\infty\left|\int_0^\infty\left\{\frac{1}{\exp(iue^{-i\delta})-1}-\frac{1}{iue^{-i\delta}}\right\}\left\{\frac{1}{\exp(-iuve^{i\delta})-1}+\frac{1}{iuve^{i\delta}}\right\}du\right|^2 dv,\end{aligned} \tag{1.3}$$

and the proof depends on approximations to integrals of the form

$$\int\frac{dz}{\{\exp(2i\pi ze^{-i\delta})-1\}\{\exp(-2i\pi kze^{i\delta})-1\}}. \tag{1.4}$$

In the previous paper I first used a method of residues, and then a method of expansion and term-by-term integration. In the new proof the method of residues is repeated. It appears at first sight as if it should be possible to extend the method to higher powers of $|\zeta(\frac{1}{2}+it)|$; but even in the next case, that of $|\zeta(\frac{1}{2}+it)|^6$, new difficulties arise, which have not been overcome.

2. Let m be an integer, $1 \leqslant m \leqslant 1/2\delta^2$, and $k > 1$. Take the integral (1.4) down the line

$$z = (m-\tfrac{1}{2}+i\lambda)e^{i\delta},$$

* A. E. Ingham, *Proc. London Math. Soc.* (2) 27 (1928), 273–300.

† E. C. Titchmarsh, ibid. (2) 27 (1928), 137–50.

where λ is real, as far as the real axis, and then along the real axis to $+\infty$. The line meets the real axis where $\lambda = -(m-\frac{1}{2})\tan\delta$, $x = (m-\frac{1}{2})\sec\delta$; and on it

$$\exp(2i\pi ze^{-i\delta}) = -e^{-2\pi\lambda}$$

and $$|\exp(-2i\pi kze^{i\delta})-1| \geqslant \exp 2\pi k\{(m-\tfrac{1}{2})\sin 2\delta+\lambda\cos 2\delta\}-1.$$

Hence the modulus of the integral along the line does not exceed

$$\int_{-(m-\frac{1}{2})\tan\delta}^{\infty} \frac{d\lambda}{\exp 2\pi k\{(m-\frac{1}{2})\sin 2\delta+\lambda\cos 2\delta\}-1}$$
$$= \frac{1}{2\pi k\cos 2\delta}\int_{2\pi k(m-\frac{1}{2})\tan\delta}^{\infty}\frac{d\mu}{e^{\mu}-1} = O\left\{\frac{1}{k}\log\left(2+\frac{1}{km\delta}\right)e^{-Akm\delta}\right\}.$$

The calculus of residues therefore gives

$$\int_{(m-\frac{1}{2})\sec\delta}^{\infty}\frac{dz}{\{\exp(2i\pi ze^{-i\delta})-1\}\{\exp(-2i\pi kze^{i\delta})-1\}}$$
$$= e^{i\delta}\sum_{r=m}^{\infty}\frac{1}{\exp(-2i\pi kre^{2i\delta})-1}+O\left\{\frac{1}{k}\log\left(2+\frac{1}{km\delta}\right)e^{-Akm\delta}\right\}. \quad (2.1)$$

Again, on the stretch $\{(m-\frac{1}{2})\sec\delta, m\}$ of the real axis,

$$|\exp(2i\pi ze^{-i\delta})-1|^2 = (e^{2\pi z\sin\delta}-1)^2+4e^{2\pi z\sin\delta}\sin^2(\pi z\cos\delta)$$
$$> A\{m^2\delta^2+(m-z)^2\},$$

and $$|\exp(-2i\pi kze^{i\delta})-1| \geqslant e^{2\pi kz\sin\delta}-1 > e^{Akm\delta}-1.$$

We may therefore replace the lower limit $(m-\frac{1}{2})\sec\delta$ by m, with error

$$O\left(\frac{1}{e^{Akm\delta}-1}\int_{m-\frac{1}{2}}^{m}\frac{dz}{\{m^2\delta^2+(m-z)^2\}^{\frac{1}{2}}}\right) = O\left(\frac{1}{e^{Akm\delta}-1}\int_{0}^{1/2m\delta}\frac{dw}{(1+w^2)^{\frac{1}{2}}}\right)$$
$$= O\left(\frac{1}{km\delta}e^{-Akm\delta}\log\frac{1}{\delta}\right).$$

3. Comparatively trivial arguments (see § 5. 1 of my paper referred to) show that the inner integral on the right-hand side of (1.3) is

$$\int_{\pi\sec\delta}^{\infty}\frac{du}{\{\exp(iue^{-i\delta})-1\}\{\exp(-iuve^{i\delta})-1\}}+O\left\{\frac{1}{v}\left(1+\log\frac{v}{\delta}\right)\right\},$$

and therefore, by (2.1) with $m = 1$, $k = v$, it is equal to

$$2\pi e^{i\delta}\sum_{r=1}^{\infty}\frac{1}{\exp(-2i\pi rve^{2i\delta})-1}+O\left\{\frac{1}{v}\left(1+\log\frac{v}{\delta}\right)\right\}.$$

Writing $$\Delta = \left[\frac{1}{\delta}\log\frac{1}{\delta}\right],$$

the terms with $r > \Delta$ are

$$O\sum_{r>\Delta} e^{-2\pi rv\sin 2\delta} = O\left(\frac{e^{-\pi\Delta v\sin 2\delta}}{v\delta}\right) = O\left(\frac{1}{v}\right).$$

The contribution of these O-terms to the result is clearly $O(\log^2 1/\delta)$.

The result therefore depends on

$$\int_1^\infty \left|\sum_{n=1}^{\Delta}\frac{1}{\exp(-2in\pi ve^{2i\delta})-1}\right|^2 dv$$

$$= \int_1^\infty \sum_{m=1}^{\Delta}\sum_{n=1}^{\Delta}\frac{dv}{\{\exp(2im\pi ve^{-2i\delta})-1\}\{\exp(-2in\pi ve^{2i\delta})-1\}}$$

$$= \sum_{m=1}^{\Delta}\int_1^\infty \frac{dv}{\{\exp(2im\pi ve^{-2i\delta})-1\}\{\exp(-2im\pi ve^{2i\delta})-1\}} +$$

$$+2\mathbf{R}\sum_{n=2}^{\Delta}\sum_{m=1}^{n-1}\int_1^\infty \frac{dv}{\{\exp(2im\pi ve^{-2i\delta})-1\}\{\exp(-2in\pi ve^{2i\delta})-1\}}.$$

We approximate to these integrals as in § 2. The O-terms arising are

$$\sum_{n=2}^{\Delta}\sum_{m=1}^{n} O\left\{\frac{1}{n}\log\left(2+\frac{1}{n\delta}\right)+\frac{1}{mn\delta}\log\frac{1}{\delta}\right\}e^{-An\delta}$$

$$= O\sum_{n=2}^{\infty}\left(\log\frac{1}{\delta}+\frac{1}{n\delta}\log^2\frac{1}{\delta}\right)e^{-An\delta} = O\left(\frac{1}{\delta}\log^3\frac{1}{\delta}\right).$$

The single m-sum gives the series

$$\sum_{m=1}^{\Delta}\frac{1}{m}\sum_{r=m}^{\infty}\frac{1}{\exp(-2i\pi re^{4i\delta})-1} = O\sum_{m=1}^{\infty}\frac{1}{m}\sum_{r=m}^{\infty}\frac{e^{-\pi r\sin 4\delta}}{r\delta}$$

$$= O\left(\frac{1}{\delta}\log\frac{1}{\delta}\sum_{m=1}^{\infty}\frac{1}{m}e^{-\pi m\sin 4\delta}\right) = O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right).$$

It therefore remains a question of the sum

$$\sum_{n=1}^{\Delta}\sum_{m=1}^{n}\frac{1}{m}\sum_{r=m}^{\infty}\frac{1}{\exp\left(-2i\pi\frac{nr}{m}e^{4i\delta}\right)-1}$$

$$= \sum_{m=1}^{\Delta}\frac{1}{m}\sum_{n=m}^{\Delta}\sum_{r=m}^{\infty}\sum_{l=1}^{\infty}\exp\left(2i\pi\frac{nrl}{m}e^{4i\delta}\right).$$

We can replace $ie^{4i\delta}$ by $i-\sin 4\delta$ with error

$$O\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}\sum_{r=m}^{\infty}\sum_{l=1}^{\infty}\frac{nrl}{m}\delta^2\exp\left(-2\pi\frac{nrl}{m}\sin 4\delta\right).$$

Since $$\sum_{l=1}^{\infty}le^{-al}=O\left\{\left(1+\frac{1}{a^2}\right)e^{-a}\right\}=O\left(\frac{1}{a^2}e^{-\frac{1}{2}a}\right)$$

for large or small a, this is

$$O\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{1}{n}\sum_{r=m}^{\infty}\frac{1}{r}\exp\left(-\pi\frac{nr}{m}\sin 4\delta\right)=O\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{1}{n}\log\frac{n}{\delta}e^{-\pi n\sin 4\delta}$$

$$=O\log^2\frac{1}{\delta}\sum_{m=1}^{\infty}e^{-\pi m\sin 4\delta}=O\left(\frac{1}{\delta}\log^2\frac{1}{\delta}\right).$$

We can also replace the upper n-limit by ∞ with error

$$O\sum_{m=1}^{\Delta}\sum_{n=\Delta}^{\infty}\sum_{r=m}^{\infty}\frac{m}{nr\delta}\exp\left(-\pi\frac{nr}{m}\sin 4\delta\right)=O\sum_{m=1}^{\Delta}\frac{m}{\delta}\log^2\frac{m}{\delta}e^{-\pi\Delta\sin 4\delta}$$

$$=O(1).$$

Hence it is a question of

$$\sum_{m=1}^{\Delta}\frac{1}{m}\sum_{n=m}^{\infty}\sum_{r=m}^{\infty}\sum_{l=1}^{\infty}\exp\left(2i\pi\frac{nrl}{m}-2\pi\frac{nrl}{m}\sin 4\delta\right).$$

4. Putting $n=\nu+\nu'm$, $r=\rho+\rho'm$, $l=\lambda+\lambda'm$, the inner triple sum is

$$\sum_{\nu=0}^{m-1}\sum_{\rho=0}^{m-1}\sum_{\lambda=1}^{m}\left[\exp\left(2i\pi\frac{\nu\rho\lambda}{m}\right)\times\right.$$

$$\left.\times\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}\sum_{\lambda'=0}^{\infty}\exp\left\{-\frac{2\pi}{m}(\nu+\nu'm)(\rho+\rho'm)(\lambda+\lambda'm)\sin 4\delta\right\}\right].$$

The terms for which $m|\nu\rho$ do not exceed

$$\sum_{\substack{\nu=0\\ m|\nu\rho}}^{m-1}\sum_{\rho=0}^{m-1}1\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}\sum_{l=1}^{\infty}\exp(-2\pi\nu'\rho'ml\sin 4\delta).$$

Let μ be a divisor of m, and let $(\nu, m)=\mu$. Corresponding to each such ν, ρ takes the values

$$0,\ \frac{m}{\mu},\ldots,\ (\mu-1)\frac{m}{\mu},$$

i.e. μ values. Since the number of such ν does not exceed m/μ, the

number of terms corresponding to each μ does not exceed m. Hence

$$\sum_{\nu=0}^{m-1}\sum_{\substack{\rho=0\\ m|\nu\rho}}^{m-1} 1 \leqslant m\,d(m).$$

Also the triple sum in ν', ρ', l is

$$O\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}\frac{1}{\nu'\rho' m\delta}e^{-\pi\nu'\rho' m\sin 4\delta} = O\sum_{\nu'=1}^{\infty}\frac{1}{\nu' m\delta}\log\frac{\nu' m}{\delta}e^{-\pi\nu' m\sin 4\delta}$$

$$= O\left(\frac{1}{m\delta}\log^2\frac{m}{\delta}e^{-\pi m\sin 4\delta}\right).$$

Hence these terms contribute

$$O\frac{1}{\delta}\sum_{m=1}^{\infty}\frac{d(m)}{m}\log^2\frac{m}{\delta}e^{-\pi m\sin 4\delta} = O\left(\frac{1}{\delta}\log^4\frac{1}{\delta}\right).$$

A similar argument applies to the terms for which $m|\nu\lambda$, except for the term $\lambda' = 0$. This gives

$$\sum_{\substack{\nu=0\\ m|\nu\lambda}}^{m-1}\sum_{\lambda=1}^{m}\sum_{\rho=0}^{m-1}1\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}\exp\left\{-\frac{2\pi}{m}(\nu+\nu' m)(\rho+\rho' m)\lambda\sin 4\delta\right\}$$

$$\leqslant m\sum_{\substack{\nu=0\\ m|\nu\lambda}}^{m-1}\sum_{\lambda=1}^{m}\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}\exp(-2\pi\nu'\rho' m\lambda\sin 4\delta)$$

$$= O\left(m\sum_{\substack{\nu=0\\ m|\nu\lambda}}^{m-1}\sum_{\lambda=1}^{m}\sum_{\nu'=1}^{\infty}\frac{1}{\nu' m\lambda\delta}e^{-\pi\nu' m\lambda\sin 4\delta}\right)$$

$$= O\left(\frac{1}{\delta}\log\frac{m}{\delta}e^{-\pi m\sin 4\delta}\sum_{\substack{\nu=0\\ m|\nu\lambda}}^{m-1}\sum_{\lambda=1}^{m}\frac{1}{\lambda}\right).$$

Defining μ and ν as before, λ takes the values

$$\frac{m}{\mu},\ \frac{2m}{\mu},\ldots,\ (\mu-1)\frac{m}{\mu},$$

and

$$\sum\frac{1}{\lambda} = \sum_{k=1}^{\mu=1}\frac{\mu}{km} = O\left(\frac{\mu\log\mu}{m}\right) = O\left(\frac{\mu\log m}{m}\right).$$

Hence the sum of terms corresponding to each μ is $O(\log m)$, and

$$\sum_{\substack{\nu=0\\ m|\nu\lambda}}^{m-1}\sum_{\lambda=1}^{m}\frac{1}{\lambda} = O\{d(m)\log m\}.$$

The same result now follows for this sum.

5. Next consider the terms for which $\lambda' > 0$, $m \nmid \nu\rho$. Here

$$\sum_{\lambda=1}^{m} \exp\left\{2i\pi\frac{\nu\rho\lambda}{m} - 2\pi\frac{(\nu+\nu'm)(\rho+\rho'm)}{m}\lambda\sin 4\delta\right\}$$

$$= \frac{1-\exp\{2i\pi\nu\rho - 2\pi(\nu+\nu'm)(\rho+\rho'm)\sin 4\delta\}}{1-\exp\left\{2i\pi\frac{\nu\rho}{m} - \frac{2\pi}{m}(\nu+\nu'm)(\rho+\rho'm)\sin 4\delta\right\}} \times$$

$$\times \exp\left\{2i\pi\frac{\nu\rho}{m} - \frac{2\pi}{m}(\nu+\nu'm)(\rho+\rho'm)\sin 4\delta\right\}$$

$$= O\left[\exp\left\{-\frac{2\pi}{m}(\nu+\nu'm)(\rho+\rho'm)\sin 4\delta\right\}\Big/\left(\frac{\nu\rho}{m}\right)\right],$$

where (x) denotes the distance of x from the nearest integer. We therefore obtain

$$O\left\{\sum_{\nu=0}^{m-1}\sum_{\substack{\rho=0\\ m\nmid\nu\rho}}^{m-1}\frac{1}{(\nu\rho/m)}\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}\sum_{\lambda'=1}^{\infty}e^{-2\pi\nu'\rho'\lambda'm^2\sin 4\delta}\right\}.$$

If $\nu/m = p/q$ in its lowest terms, $(\nu\rho/m)$ takes some of the values $1/q$, $2/q$,..., $(q-1)/q$, each at most $2m/q$ times. Hence the ρ-sum is

$$O\left(\frac{m}{q}\sum_{k=1}^{q-1}\frac{q}{k}\right) = O(m\log q) = O(m\log m).$$

The whole sum is therefore

$$O\left(m^2\log m\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}\frac{1}{\nu'\rho'm^2\delta}e^{-\pi\nu'\rho'm^2\sin 4\delta}\right) = O\left(\frac{\log m}{\delta}\log^2\frac{m}{\delta}e^{-\pi m^2\sin 4\delta}\right),$$

and this gives

$$\frac{1}{\delta}\sum_{m=1}^{\infty}\frac{1}{m}\left(\log^3 m+\log m\log^2\frac{1}{\delta}\right)e^{-\pi m^2\sin 4\delta} = O\left(\frac{1}{\delta}\log^4\frac{1}{\delta}\right).$$

Finally, if $\lambda' = 0$, $m \nmid \nu\rho$, we get

$$\sum_{\nu=0}^{m-1}\sum_{\lambda=1}^{m}\sum_{\substack{\nu'=1\\ m\nmid\nu\lambda}}^{\infty}\sum_{\rho'=1}^{\infty}\sum_{\rho=0}^{m-1}\exp\left\{2i\pi\frac{\nu\rho\lambda}{m} - \frac{2\pi}{m}(\nu+\nu'm)(\rho+\rho'm)\lambda\sin 4\delta\right\}$$

$$= O\left\{\sum_{\substack{\nu=0\\ m\nmid\nu\lambda}}^{m-1}\sum_{\lambda=1}^{m}\frac{1}{(\nu\lambda/m)}\sum_{\nu'=1}^{\infty}\sum_{\rho'=1}^{\infty}e^{-2\pi\nu'\rho'm\lambda\sin 4\delta}\right\}$$

$$= O\left\{\sum_{\lambda=1}^{m}m\log m\frac{1}{m\lambda\delta}\log\frac{m\lambda}{\delta}e^{-\pi m\sin 4\delta}\right\}$$

$$= O\left\{\frac{1}{\delta}\left(\log^3 m+\log^2 m\log\frac{1}{\delta}\right)e^{-\pi m\sin 4\delta}\right\}.$$

Hence this part also gives $O\left(\frac{1}{\delta}\log^4\frac{1}{\delta}\right)$; this completes the proof of (1.2).

A NEW PROOF OF DIXON'S THEOREM ON HYPERGEOMETRIC SERIES

By W. N. BAILEY (*Manchester*)

[Received 30 December 1936]

1. THERE have been published three proofs of Dixon's theorem

$$ {}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c & \end{matrix}\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}. \quad (1.1) $$

The original proof given by Dixon* involved the use of integrals of Pochhammer's type, and was long and involved. The second proof was given by Dougall† and independently by Ramanujan,‡ both of whom deduced the theorem by a limiting process from Dougall's theorem for the sum of a well-poised terminating series of the type ${}_7F_6$. A simpler and more direct proof was given by Watson§ who used double series together with the theorems of Gauss and Kummer for summing the ordinary hypergeometric series with arguments ± 1.

All these proofs are quite different from one of the standard methods‖ of deducing Gauss's theorem from a relation between contiguous hypergeometric series of the ordinary type. In this paper I give an analogous method of proving Dixon's theorem.

2. We begin with the formula

$$ \begin{aligned} ab(1-x){}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1; \\ 1+a-b, & 1+a-c & \end{matrix} x\right] &= (a-2c)(a-b-c){}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c & \end{matrix} x\right] - \\ &\quad -(a-c)(a-2b-2c){}_3F_2\left[\begin{matrix} a, & b, & c+1; \\ 1+a-b, & a-c & \end{matrix} x\right] \end{aligned} \quad (2.1) $$

which can be proved very easily by comparing the coefficients of x^r

* A. C. Dixon, *Proc. London Math. Soc.* (1), 35 (1903), 284–9.

† J. Dougall, *Proc. Edinburgh Math. Soc.* 25 (1907), 114–32, (14).

‡ G. H. Hardy, *Proc. Cambridge Phil. Soc.* 21 (1923), 492–503, (3.2).

§ G. N. Watson, *Proc. London Math. Soc.* (2), 22 (1924), xxxii–xxxiii.

‖ See, for instance, Whittaker and Watson, *Modern Analysis*, § 14.53.

on the two sides. Now make $x \to 1$, and change c into $c-1$, and we get

$$
{}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c & \end{matrix}\right]
= \frac{(1+\frac{1}{2}a-c)(1+a-b-c)}{(1+a-c)(1+\frac{1}{2}a-b-c)}\,{}_3F_2\left[\begin{matrix} a, & b, & c-1; \\ 1+a-b, & 2+a-c & \end{matrix}\right], \quad (2.2)
$$

provided that the series on the left is convergent.*

By repeating (2.2) we obtain

$$
{}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c & \end{matrix}\right]
= \frac{(1+a-b-c)_n(1+\frac{1}{2}a-c)_n}{(1+a-c)_n(1+\frac{1}{2}a-b-c)_n}\,{}_3F_2\left[\begin{matrix} a, & b, & c-n; \\ 1+a-b, & 1+a-c+n & \end{matrix}\right],
$$

and, making $n \to \infty$, we get

$$
{}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c & \end{matrix}\right]
= \frac{\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a-b-c)\Gamma(1+\frac{1}{2}a-c)}\,{}_2F_1\left[\begin{matrix} a, & b; \\ 1+a-b & \end{matrix} -1\right],
$$

provided that the series on the right converges. This series can be summed by Kummer's theorem and the result follows.

The proof requires that $R(b) < 1$, but this condition can be removed by an appeal to the theory of analytic continuation.

3. The formula (2.1) is a particular case of the formula

$$
\begin{aligned}
ab(1-x){}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1; \\ d, & e & \end{matrix}\, x\right]
&= (d-c-1)(e-c-1){}_3F_2\left[\begin{matrix} a, & b, & c; \\ d, & e & \end{matrix}\, x\right] - \\
&-(d+e-a-b-c-2)(e-1){}_3F_2\left[\begin{matrix} a, & b, & c+1; \\ d, & e-1 & \end{matrix}\, x\right] + \\
&+\{(e-a-1)(e-b-1)-c(c-d+1)\}{}_3F_2\left[\begin{matrix} a, & b, & c+1; \\ d, & e & \end{matrix}\, x\right], \quad (3.1)
\end{aligned}
$$

which reduces to (2.1) when $d = 1+a-b$, $e = 1+a-c$. When $d = c+1$, (3.1) reduces to one of the relations between associated hypergeometric series of the ordinary type, from which Gauss's theorem can be deduced by the usual method.

* When $x \to 1$, it is evident that the left-hand side of (2.1) tends to zero when $R(a-2b-2c) > 1$, in which case the ${}_3F_2$ converges for $x = 1$. The proof that this is still true under the wider condition $R(a-2b-2c) > 0$ is identical with the argument used in the proof of Gauss's theorem.

ASSOCIATED HYPERGEOMETRIC SERIES

By W. N. BAILEY (*Manchester*)

[Received 30 December 1936]

1. THE main purpose of this note is to give new and simple proofs of the formulae

$$(a)_n x^{a-1}F(a+n,b;c;x) = D^n[x^{a+n-1}F(a,b;c;x)], \quad (1.1)$$

$$(c-n)_n x^{c-1-n}F(a,b;c-n;x) = D^n[x^{c-1}F(a,b;c;x)], \quad (1.2)$$

$$(a)_n(b)_n F(a+n,b+n;c+n;x) = (c)_n D^n[F(a,b;c;x)], \quad (1.3)$$

$$(c-a)_n x^{c-a-1}(1-x)^{a+b-c-n}F(a-n,b;c;x) = D^n[x^{c-a+n-1}(1-x)^{a+b-c}F(a,b;c;x)], \quad (1.4)$$

$$(c-n)_n x^{c-1-n}(1-x)^{a+b-c-n}F(a-n,b-n;c-n;x) = D^n[x^{c-1}(1-x)^{a+b-c}F(a,b;c;x)], \quad (1.5)$$

$$(c-a)_n(c-b)_n(1-x)^{a+b-c-n}F(a,b;c+n;x) = (c)_n D^n[(1-x)^{a+b-c}F(a,b;c;x)], \quad (1.6)$$

$$(-1)^n(a)_n(c-b)_n(1-x)^{a-1}F(a+n,b;c+n;x) = (c)_n D^n[(1-x)^{a+n-1}F(a,b;c;x)], \quad (1.7)$$

$$(c-n)_n x^{c-1-n}(1-x)^{b-c}F(a-n,b;c-n;x) = D^n[x^{c-1}(1-x)^{b-c+n}F(a,b;c;x)], \quad (1.8)$$

where $D = d/dx$.

The formula (1.5) is a well-known result due to Jacobi.* The formulae (1.1) to (1.6), apart from (1.3) and (1.5), were given recently by Poole,† and (1.7) and (1.8), the latter in an incorrect form, were given as a postscript to the same paper.

In this paper I give extremely simple proofs of the first six formulae, and rather less simple proofs of the last two.

2. The proofs of (1.1)–(1.3) simply consist in expanding $F(a,b;c;x)$ in powers of x and performing the differentiations term by term. Now transform (1.1) by using the formula

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x), \quad (2.1)$$

and we obtain

$$(a)_n x^{a-1}(1-x)^{c-a-b-n}F(c-a-n,c-b;c;x) = D^n[x^{a+n-1}(1-x)^{c-a-b}F(c-a,c-b;c;x)],$$

* C. G. J. Jacobi, *Werke*, vi. 191–3. It is more usual to write the formula with $a+n$, $b+n$, $c+n$ instead of a, b, c.

† E. G. C. Poole, *Quart. J. of Math.* (Oxford), 6 (1935), 214–16.

and this is simply (1.4) with an obvious change of notation. The formulae (1.5) and (1.6) can be derived from (1.2) and (1.3) in a similar manner.

Now write $x = 1-y$, and denote d/dy by D_1. Then, using the formula expressing $F(a,b;c;x)$ in terms of functions with argument $1-x$, we have

$$D^n[(1-x)^{a+n-1}F(a,b;c;x)]$$
$$= (-1)^n D_1^n\Big[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}y^{a+n-1}F(a,b;a+b+1-c;y)+$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}y^{c-b+n-1}F(c-a,c-b;1+c-a-b;y)\Big],$$

and by (1.1) this is equal to

$$(-1)^n\Big[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}(a)_n y^{a-1}F(a+n,b;c;y)+$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(c-b)_n y^{c-b-1}F(c-b+n,c-a;1+c-a-b;y)\Big]$$
$$= \frac{(-1)^n(a)_n(c-b)_n}{(c)_n}(1-x)^{a-1}F(a+n,b;c+n;x),$$

and this proves (1.7). Similarly we can derive (1.8) from (1.4).

3. Instead of using (2.1) we can use other transformations of hypergeometric series, for example

$$F(a,b;c;x) = (1-x)^{-a}F\{a,c-b;c;-x/(1-x)\}. \tag{3.1}$$

Using this with (1.1) we get

$$(a)_n x^{a-1}(1-x)^{-a-n}F\{a+n,c-b;c;-x/(1-x)\}$$
$$= D^n[x^{a+n-1}(1-x)^{-a}F\{a,c-b;c;-x/(1-x)\}],$$

or, with a change of notation,

$$(a)_n x^{a-1}(1-x)^{n+1}F(a+n,b;c;x)$$
$$= \{(1-x)^2D\}^n[x^{a+n-1}(1-x)^{1-n}F(a,b;c;x)]. \tag{3.2}$$

Similarly from (1.2) and (1.3) used with (3.1) we derive the formulae

$$(c-a-n)_n x^{c-1-n}(1-x)^{b-c+1+n}F(a-n,b;c-n;x)$$
$$= \{(1-x)^2D\}^n[x^{c-1}(1-x)^{b-c+1}F(a,b;c;x)], \tag{3.3}$$
$$(a)_n(c-b)_n(1-x)^{a+n}F(a+n,b;c+n;x)$$
$$= (-1)^n(c)_n\{(1-x)^2D\}^n[(1-x)^aF(a,b;c;x)], \tag{3.4}$$

and from (3.2) and (2.1) we obtain

$$(c-a)_n x^{c-a-1}(1-x)^{a+b-c+1}F(a-n,b;c;x)$$
$$= \{(1-x)^2D\}^n[x^{c-a+n-1}(1-x)^{a+b-c+1-n}F(a,b;c;x)]. \quad (3.5)$$

In the same way, by using (3.1) with (1.7) and (1.8), we derive the formulae

$$(a)_n(b)_n(1-x)^{n+1}F(a+n,b+n;c+n;x)$$
$$= (c)_n\{(1-x)^2D\}^n[(1-x)^{1-n}F(a,b;c;x)], \quad (3.6)$$

$$(c-a)_n(c-b)_n(1-x)^{a+b-c+1}F(a,b;c+n;x)$$
$$= (c)_n\{(1-x)^2D\}^n[(1-x)^{a+b-c+1-n}F(a,b;c;x)], \quad (3.7)$$

$$(c-n)_n x^{c-1-n}(1-x)^{n+1}F(a,b;c-n;x)$$
$$= \{(1-x)^2D\}^n[x^{c-1}(1-x)^{1-n}F(a,b;c;x)], \quad (3.8)$$

$$(c-n)_n x^{c-1-n}(1-x)^{a+b-c+1}F(a-n,b-n;c-n;x)$$
$$= \{(1-x)^2D\}^n[x^{c-1}(1-x)^{a+b-c+1-n}F(a,b;c;x)]. \quad (3.9)$$

4. It is natural to inquire whether any of the above formulae can be extended to series of order higher than that of the ordinary hypergeometric series. Evidently (1.1), (1.2), and (1.3) can be extended to series of any order, but there are no transformations like (2.1) and (3.1) for such series, and so we cannot expect to obtain extensions of the other formulae. We can, however, use Poole's method to obtain a generalization of Gauss's theorem (extended by Riemann), that any three series of the type $F(a+l,b+m;c+n;x)$, where l, m, n are integers (positive or negative), are connected by a linear homogeneous relation with polynomial coefficients. For example, if

$$F \equiv {}_3F_2\left[\begin{matrix}a, & b, & c;\\ d, & e\end{matrix}\ x\right], \qquad F_{a+} \equiv {}_3F_2\left[\begin{matrix}a+1, & b, & c;\\ d, & e\end{matrix}\ x\right],$$

and $\delta \equiv xd/dx$, it is easily shown that

$$aF_{a+} = (\delta+a)F, \quad (4.1)$$

$$(e-1)F_{e-} = (\delta+e-1)F, \quad (4.2)$$

$$(d-a)(e-a)F_{a-}$$
$$= [\{\delta^2+\delta(d+e-a-1)+(d-a)(e-a)\}-x(\delta+b)(\delta+c)]F, \quad (4.3)$$

$$x\{abc-e^2(a+b+c)+e^3\}F_{e+}$$
$$= e[\delta(\delta+d-1)-x\{\delta^2+\delta(a+b+c-e)+bc+ca+ab-$$
$$-e(a+b+c)+e^2\}]F. \quad (4.4)$$

By repeated applications of these and similar formulae, and by use of the differential equation satisfied by F, it is evident that the associated series

$$F\left[\begin{matrix} a+l, & b+m, & c+n; \\ d+p, & e+q & \end{matrix} \; x\right],$$

where l, m, n, p, q are integers, can be expressed in terms of F, δF, $\delta^2 F$. From four relations of this type we can eliminate $F, \delta F, \delta^2 F$, and we see that between any four associated series of the type ${}_3F_2$ there is a linear homogeneous relation with polynomial coefficients. A particular case of one such relation* is used in the preceding paper to obtain a new proof of Dixon's theorem.

* p. 116 (3.1).

A TYPE OF 'MAGIC SQUARE' IN TARRY'S PROBLEM

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[Received 19 March 1937]

1. Introduction

TARRY'S problem has been recently discussed in this *Journal* by E. M. Wright (**1, 2, 3**), who gives the necessary references (**1**). Of these the most germane to this paper is Dickson (**4**). For our present purposes the problem consists essentially in the solution, in integers,* of the system of equations

$$\sum_{r=1}^{n} x_r^k = \sum_{r=1}^{n} y_r^k \quad (k = 1, 2, \ldots, m). \tag{1}$$

Wright discusses, for given m, the least n for which an integer solution of these equations can be found. Now, the equations imply (and are implied by) the equivalence of the corresponding elementary symmetric functions in x and y respectively of orders up to m inclusive. Evidently m cannot be as great as n, for then *every* such function of x would be equal to the corresponding function of y, so that (x_r) and (y_r) would each be the set of roots of the same n-ic, i.e. would be merely a rearrangement of each other. This is rejected as a trivial solution. Thus, for given n, the greatest m than can give a non-trivial solution is $m = n-1$. This *optimum*† case is the only one we shall consider.

In this optimum case the x's and y's are respectively the solutions of two n-ics.

$$X(x) = 0, \qquad Y(y) = 0$$

which differ only in their constant terms,

i.e.

$$X(t) - Y(t) = c, \quad \text{say.}$$

Hence, for every r and every s,

$$X(y_s) = c, \qquad Y(x_r) = -c,$$

* Or in positive integers. For the reason given at the end of this section we can always convert an integer solution into a positive-integer solution.

† Wright takes m as his fundamental number, whereas we have taken n, so that his minimum is our maximum, and we have therefore chosen the term 'optimum'.

i.e.

$$\left.\begin{aligned}\prod_{r=1}^{n}(x_r-y_s) &= -c\\ \prod_{s=1}^{n}(x_r-y_s) &= (-)^n c\end{aligned}\right\}. \tag{2}$$

Thus the square array formed of the differences x_r-y_s has the properties:

(i) *the product of the elements in each row* (*column*) *is the same, row-product and column-product being equal if* n *is odd, or equal with changed sign if* n *is even;*

(ii) *in any two rows* (*columns*) *the algebraic difference of the elements occupying corresponding places is constant for those rows* (*columns*).

In view of these properties we have allowed ourselves to call such a square a 'magic square'. Evidently, then, every solution of Tarry's equations in the optimum case

$$\sum_{r=1}^{n} x_r^k = \sum_{r=1}^{n} y_r^k \quad (k = 1, 2, \ldots, n-1) \tag{3}$$

gives such a 'magic square', and conversely any such square gives a solution of the equations (3).

In such a square the elements are not all of one sign, since, for example, the first equation of (3) shows that the sum of the elements in the principal diagonal vanishes. We see too that the solution is essentially given in terms of the *differences* only of x, y: this is right, since Tarry's equations (1), if they have a solution (x_r, y_s), have also the more general solution (x_r+h, y_s+h) for any h.*

2. The symmetric solutions

The possibility of two specially symmetric solutions should be noted. If n is odd, we may, by imposing the supplementary conditions

$$y_r = -x_r \quad (r = 1, \ldots, n), \tag{4}$$

reduce the equations (3) to the system

$$\sum_{r=1}^{n} x_r^k = 0 \quad (k = 1, 3, \ldots, n-2). \tag{5}$$

The pair of elements x_r-y_s, x_s-y_r are then equal, and the square is

* For this reason a solution in *positive integers* does not differ essentially from a solution in *integers*, since we can always choose h large enough to make every x, y positive. For some purposes it might be advantageous to define a 'standard' solution in which h had been chosen so that $\Sigma x = 0 = \Sigma y$.

axi-symmetric. If n is even, the supplementary conditions

$$x_{n+1-r} = -x_r, \qquad y_{n+1-r} = -y_r \quad (r = 1, \ldots, \tfrac{1}{2}n), \tag{6}$$

reduce the equations (3) to the system

$$\sum_1^{n/2} x_r^k = \sum_1^{n/2} y_r^k \quad (k = 2, 4, \ldots, n-2). \tag{7}$$

The pair of elements $x_r - y_s$, $x_{n+1-r} - y_{n+1-s}$, which are images in the centre of the square, are then equal but of opposite sign, and the square is *centrally skew*. Most of the known solutions of Tarry's problem correspond to one or other of these symmetric forms.

The property (i) of the magic square shows that the same prime factors must occur, differently distributed, in each column and in each row. For purely numerical solutions of equations (3) the law of this distribution is not very apparent and, in the hope of further elucidation, we have considered more especially solutions of the equations (3), which express x, y as rational (integral) algebraic functions of suitable parameters.* In particular, we have looked for a *general* algebraic solution, i.e. a solution obtainable by deduction from the original equations, which must therefore embrace all possible solutions. Parametric solutions of the 'optimum' case appear in the literature for $n = 2, 3, 4, 6$, and numerical solutions for these values of n and for $n = 5, 7, 8$. We have succeeded in obtaining parametric solutions of more or less generality for all values of n up to 8. To the natural question as to whether the method may furnish a solution of the hitherto unsolved cases $n > 8$ we must reply that it has not at present done so, and that the algebraic difficulties in the way of such solution seem considerable. The following are such results as we have been able to obtain.

3. The square of order two

For completeness' sake we begin with $n = 2$, which has the general solution

$$x_1 - y_1 = -(x_2 - y_2) = a, \quad \text{say},$$
$$x_1 - y_2 = -(x_2 - y_1) = b, \quad \text{say}.$$

* The property (i) is also convenient for the purely practical purpose of verifying an alleged solution of Tarry's equations, since the numbers to be calculated are much lower than in the direct substitution, and the multiplicity of checks shows up mere arithmetical errors.

We can thus write the corresponding square as

$$\begin{vmatrix} a & b \\ -b & -a \end{vmatrix}, \tag{8}$$

the general solution being necessarily symmetric.

4. The square of order three

For $n = 3$ we have the solution

$$\begin{vmatrix} ap & br & cq \\ bq & cp & ar \\ cr & aq & bp \end{vmatrix}, \tag{9}$$

where $$a+b+c = 0, \qquad p+q+r = 0. \tag{10}$$

We see at a glance that the product-properties (i) hold for the square (9). In virtue of (10) we have

$$ap-br = bq-cp = cr-aq,$$

and so, by the symmetry, the differences are also equal for other columns and rows. The difference-properties (ii) therefore hold for (9).

It can be shown that (9) is the general square, for, writing

$$(rs) = x_r - y_s, \tag{11}$$

we must have in any square

$$(11)(12)(13) = (11)(21)(31),$$

since, by (i), product of first row = product of first column,

i.e. $$(12)(13) = (21)(31).$$

We can thus write

$$(12) = br, \qquad (13) = cq, \qquad (21) = bq, \qquad (31) = cr.$$

Then, defining a, p as in (10), we have, since $\sum x = \sum y$,

$$(22) = -(13)-(31) = cp,$$

$$(33) = -(12)-(21) = bp,$$

$$(11) = -(22)-(33) = ap.$$

Finally,

$$(23) = (22)+(13)-(12) = aq,$$

$$(32) = (22)+(31)-(21) = ar.$$

The axi-symmetric solution is deduced (if required) by writing $2q = 2r = -p$: it is

$$\begin{array}{|ccc|} \hline 2a & -b & -c \\ -b & 2b & -a \\ -c & -a & 2c \\ \hline \end{array} . \tag{12}$$

5. A square of order six

From (9) we can construct a square for $n = 6$ in the form

$$\begin{array}{|ccc:ccc|} \hline a(q-r) & b(r-p) & c(p-q) & (b-a)r & (a-c)q & (c-b)p \\ b(p-q) & c(q-r) & a(r-p) & (c-b)q & (b-a)p & (a-c)r \\ c(r-p) & a(p-q) & b(q-r) & (a-c)p & (c-b)r & (b-a)q \\ \hdashline (a-b)q & (b-c)r & (c-a)p & b(r-q) & a(q-p) & c(p-r) \\ (c-a)r & (a-b)p & (b-c)q & a(p-r) & c(r-q) & b(q-p) \\ (b-c)p & (c-a)q & (a-b)r & c(q-p) & b(p-r) & a(r-q) \\ \hline \end{array} \tag{13}$$

where $$a+b+c = 0, \qquad p+q+r = 0. \tag{14}$$

The product-properties hold, since they hold for the constituent quarter-squares. The difference-properties hold for each half-column and half-row: we have only to show that the differences are the same for the two halves of each column, and so for the rows. For the first two columns these differences are

$$a(q-r)-b(r-p), \qquad (a-b)q-(b-c)r,$$

which are equal in virtue of (14). It follows by symmetry that the other corresponding 'half-differences' will be equal. This square (13) is symmetric, but is (presumably) not the most general symmetric square of order six, for, from its construction, the solution satisfies the condition

$$(x_1-y_1)+(x_2-y_2)+(x_3-y_3) = 0, \tag{15}$$

which is not deducible from the symmetric equations (3).

We have been unable to solve the equations (3) for $n = 6$ without attaching to them a restrictive condition equivalent to (15). Moreover, we have failed to extend to squares of higher order this device of constructing a square of order $2n$ out of suitable squares of order n.

6. The square of order four

For $n = 4$ we have the square

$$\begin{array}{|cccc|}
p(q+r)(ps-qr) & -qr(p+s)\sigma & -s(p+r)(qs-pr) & -(q+s)\Sigma \\
qs(p+r)\sigma & p(q+s)(qs-pr) & (q+r)\Sigma & -r(p+s)(ps-qr) \\
r(q+s)(qs-pr) & -(p+r)\Sigma & -q(p+s)(ps-qr) & -ps(q+r)\sigma \\
(p+s)\Sigma & s(q+r)(ps-qr) & pr(q+s)\sigma & -q(p+r)(qs-pr)
\end{array}, \tag{16}$$

where $$\sigma \equiv p+q+r+s, \qquad \Sigma \equiv qrs+rsp+spq+pqr. \tag{17}$$

This is the *general* solution; for use the notation (11) and write R_r, C_s for the products of the rth row and the sth column. Then $R_1 R_3 = C_2 C_4$ gives

$$(11)(13)(31)(33) = (22)(24)(42)(44),$$

and we can therefore write

$$(11) = p\alpha, \qquad (22) = p\beta, \qquad (33) = q\gamma, \qquad (44) = q\delta,$$
$$(13) = s\delta, \qquad (24) = r\gamma, \qquad (31) = r\beta, \qquad (42) = s\alpha.$$

From

$$(11)+(33)+(42)+(24) = 0, \qquad (22)+(31)+(44)+(13) = 0$$

we obtain

$$(p+s)\alpha+(q+r)\gamma = 0, \qquad (p+r)\beta+(q+s)\delta = 0,$$

so that, for some λ, μ,

$$\alpha = (q+r)\lambda, \quad \beta = (q+s)\mu, \quad \gamma = -(p+s)\lambda, \quad \delta = -(p+r)\mu.$$

But $$(11)+(22)+(33)+(44) = 0,$$

and so $$p(\alpha+\beta)+q(\gamma+\delta) = 0,$$

i.e. $$\lambda(pr-qs)+\mu(ps-qr) = 0.$$

Hence, sufficiently, we may write

$$\lambda = ps-qr, \qquad \mu = qs-pr.$$

This completes the determination of half the elements of the square, let us say those 'on the black squares'. To determine the elements on the 'white squares', substitute for the known elements in $R_1 = -C_4$, $R_4 = -C_1$. We can then write, for some θ, ϕ,

$$(12) = \theta qr(p+s), \qquad (34) = \theta ps(q+r),$$
$$(21) = \phi qs(p+r), \qquad (43) = \phi pr(q+s).$$

But $$(12)+(21)+(34)+(43) = 0$$

and so $$\theta+\phi = 0.$$

Thus $$(12)+(21) = \theta pq(r-s).$$

Again, $$(12)+(21) = (11)+(22) = -pq(r-s)\sigma,$$

and so $\theta = -\sigma$. This gives (12), (21), (34), (43) and we complete the square from the difference-properties.

If in (16) we write $p = q$, we get a symmetric square of order four.

7. The square of order five

The axi-symmetric square of order five can be shown in the form

$$\begin{array}{|ccccc|}
\hline
2pP & (ps-qr)Q & (p-q)sR & -(r-s)S & qrT \\
(ps-qr)Q & -2q(r-s)R & rS & -psT & -(p-q)P \\
(p-q)sR & rS & 2(ps-qr)T & -qP & p(r-s)Q \\
-(r-s)S & -psT & -qP & -2(p-q)rQ & -(ps-qr)R \\
qrT & -(p-q)P & p(r-s)Q & -(ps-qr)R & -2sS \\
\hline
\end{array}, \tag{18}$$

where, for brevity, we have written

$$\left.\begin{array}{ll}
P \equiv ps^2-2qrs+qr^2, & Q \equiv ps-qr+qs \\
S \equiv p^2s-2pqr+q^2r, & R \equiv ps-qr+pr \\
\multicolumn{2}{c}{T \equiv pr-qs}
\end{array}\right\}. \tag{19}$$

It is readily seen that (18) is the *general* axi-symmetric square of order five. For the appropriate equations (5) are now

$$\sum_{r=1}^{5} x_r = 0, \qquad \sum_{r=1}^{5} x_r^3 = 0. \tag{20}$$

In the first of these we can put

$$\left.\begin{array}{l}
\dfrac{x_1}{p} = \dfrac{x_3+x_4}{-q} = \dfrac{x_2+x_5}{q-p} \\
\dfrac{x_5}{s} = \dfrac{x_2+x_3}{-r} = \dfrac{x_1+x_4}{r-s}
\end{array}\right\}. \tag{21}$$

We can use these equations to express x_2, x_3, x_4 in terms of x_1, x_5, and then substitute in the second equation of (20). After reduction, this gives, in the notation of (19),

$$3sSx_1^2x_5+3pPx_1x_5^2 = 0,$$

so that, rejecting $x_1 = 0 = x_5$, we may sufficiently write

$$x_1 = pP, \qquad x_5 = -sS.$$

The values of x_2, x_3, x_4 follow from (21) as

$$x_2 = -q(r-s)R, \qquad x_4 = -r(p-q)Q,$$
$$x_3 = (ps-qr)T,$$

and the square (18) can now be constructed.

8. A second square of order five

We have also been able to construct a restricted square of order five, less symmetric than (18). To do this we write

$$x_1 = -y_1, \qquad x_5 = -y_5 \tag{22}$$

in the general equations (3). The equations corresponding to $k = 2, 4$ then become

$$\sum_{r=2}^{4} x_r^2 = \sum_{r=2}^{4} y_r^2, \qquad \sum_{r=2}^{4} x_r^4 = \sum_{r=2}^{4} y_r^4, \tag{23}$$

which are the defining equations, similar to (7), of the symmetric (centrally skew) square of order 6. For the solution of these equations (23) we use the most general form we know, namely that represented by the square (13). The signs of x_r, y_r are undefined by (23), and we assign them in such a way that we find, in virtue of (13), that

$$x_2+x_4 = x_3, \qquad y_2+y_4 = y_3. \tag{24}$$

Alternatively, we could solve (3) for the case $n = 5$ by imposing on them (22) and (24) as 'reducing' conditions. In either case we at length reach the parametric solution*

$$\left.\begin{aligned} x_1 &= -pq(r^2+rs-2s^2)+q^2r(r-s) \\ x_2 &= -p^2s(r-s)+pq(r^2-2rs-s^2)-q^2r(r+s) \\ x_3 &= p^2r(r-s)-2pqs(r+s)-q^2r(r-s) \\ x_4 &= p^2(r^2-s^2)-pq(r^2+s^2)+2q^2rs \\ x_5 &= p^2s(r-s)+pqs(r+s)-2q^2rs \end{aligned}\right\}, \tag{25}$$

$$\left.\begin{aligned} y_1 &= pq(r^2+rs-2s^2)-q^2r(r-s) \\ y_2 &= p^2r(r-s)+2pqs^2-q^2r(r+s) \\ y_3 &= p^2(r^2-s^2)-pq(r^2+2rs-s^2)-2q^2rs \\ y_4 &= p^2s(r-s)-pq(r+s)^2+q^2r(r-s) \\ y_5 &= -p^2s(r-s)-pqs(r+s)+2q^2rs \end{aligned}\right\}. \tag{26}$$

From (25), (26) we construct the square as

$-2q(r-s)P$	$-r(p-q)T$	$-(p+q)(r+s)R$	$-psS$	$(ps-qr)Q$
$-s(p+q)T$	$-p(r+s)Q$	$-(p-q)(r-s)P$	$-2(ps-qr)R$	qrS
prS	$-2qsP$	$-(ps-qr)Q$	$(p-q)(r-s)T$	$(p+q)(r+s)R$
$(p-q)(r+s)R$	$(ps-qr)S$	$2qsT$	$(p+q)rQ$	$p(r-s)P$
$(ps-qr)Q$	$-(p+q)(r-s)R$	$-prS$	$q(r+s)P$	$2s(p-q)T$

, (27)

* The p, q, r of (25), (26), (27) are not, of course, the p, q, r of (13). Here p, q, r, s are all independent, the ratios p/q, r/s forming two distinct *absolute* parameters.

where now we have written

$$\left.\begin{array}{c} P \equiv R+3ps, \quad Q \equiv R+3qs, \quad S \equiv R-3qs, \quad T \equiv R+3qr \\ R = pr-ps-qr \end{array}\right\}. \tag{28}$$

We may note that the square has a certain symmetry about its centre, in that interchange of p with r, and q with s also interchanges suffixes 2, 4, and, with change of sign, suffixes 1, 5. We may also note in the square that, in the notation (11),

$$(31) = -(53), \qquad (13) = -(35),$$

and so

$$(32)(34) = (23)(43), \qquad (21)(25)(41)(45) = (12)(52)(14)(54).$$

These relations offer an alternative method of obtaining (27) along lines similar to those used in § 6 for the square of order four.

For the square of order six we have no solution other than that given above in (13).

9. The square of order seven

For the square of order seven we have nothing more general than an axi-symmetric solution depending on a single absolute parameter. To obtain this we write

$$\left.\begin{array}{c} x_1 = \tfrac{1}{2}(v+w-u), \quad x_2 = \tfrac{1}{2}(w+u-v), \quad x_3 = \tfrac{1}{2}(u+v-w), \\ x_4+x_5+x_6 = 0. \end{array}\right\} \tag{29}$$

It is then found that the defining equations (5) of the axi-symmetric solution are satisfied if

$$x_4x_5x_6 = uvw, \qquad x_4^2+x_5^2+x_6^2 = u^2+v^2+w^2. \tag{30}$$

We satisfy the first of these by writing

$$qx_4 = ru, \qquad rx_5 = pv, \qquad px_6 = qw.$$

The second then requires

$$x_4^2p^2q^2(q^2-r^2)+x_5^2q^2r^2(r^2-p^2)+x_6^2r^2p^2(p^2-q^2) = 0. \tag{31}$$

This is clearly satisfied by

$$x_4^2 = r^2, \qquad x_5^2 = p^2, \qquad x_6^2 = q^2.$$

Without loss of generality we can write

$$x_4 = r, \qquad x_5 = p, \qquad x_6 = q,$$

so that (29) is also satisfied if we impose on p, q, r the condition

$$p+q+r = 0. \tag{32}$$

This leads to a trivial solution, but the other solution of the quadratic system (29), (31) is now rational. This other solution at length gives

$$\left.\begin{aligned} x_1 &= rp(r-q)(p^2+qr), & x_4 &= -r(p^4-2p^2q^2+q^2r^2) \\ x_2 &= pq(p-r)(q^2+rp), & x_5 &= -p(q^4-2q^2r^2+r^2p^2) \\ x_3 &= qr(q-p)(r^2+pq), & x_6 &= -q(r^4-2r^2p^2+p^2q^2) \\ & x_7 = -(qr+rp+pq)(qr^2+rp^2+pq^2) \end{aligned}\right\}. \tag{33}$$

If we take $p, q, r = 2, 1, -3$, we get the solution 24, −50, 33, 51, −38, −13, −7 originally due to Eric Phillips.*

The corresponding square is

$-2rp(q-r)P$	$-pR'$	$-rQ'$	$-r(r-p)(p-q)Q$
$-pR'$	$-2pq(r-p)Q$	$-qP'$	$-\theta P$
$-rQ'$	$-qP'$	$-2qr(p-q)R$	rp^2S
$-r(r-p)(p-q)Q$	$-\theta P$	rp^2S	$-2rP'$
pq^2S	$-p(p-q)(q-r)R$	$-\theta Q$	qR'
$-\theta R$	qr^2S	$-q(q-r)(r-p)P$	pQ'
qP'	rQ'	pR'	$q^2(q-r)R$

pq^2S	$-\theta R$	qP'
$-p(p-q)(q-r)R$	qr^2S	rQ'
$-\theta Q$	$-q(q-r)(r-p)P$	pR'
qR'	pQ'	$q^2(q-r)R$
$-2pQ'$	rP'	$r^2(r-p)P$
rP'	$-2qR'$	$p^2(p-q)Q$
$r^2(r-p)P$	$p^2(p-q)Q$	$-2\theta S$

, (34)

where now we have written

$$\left.\begin{aligned} P \equiv p^2+qr, \quad Q \equiv q^2+rp, \quad R \equiv r^2+pq \\ P' \equiv p^4-2p^2q^2+q^2r^2, \; Q' \equiv q^4-2q^2r^2+r^2p^2, \; R' \equiv r^4-2r^2p^2+p^2q^2 \\ S \equiv qr+rp+pq, \quad \theta \equiv qr^2+rp^2+pq^2 \end{aligned}\right\}, \tag{35}$$

and, of course, $$p+q+r = 0.$$

It is possible to devise other rational solutions of the quadratic system (29), (31), but they give less simple alternatives to (33), (34).

We may notice too that (31) is also satisfied by $x_4^2 = q^{-2}$, $x_5^2 = r^{-2}$, $x_6^2 = p^{-2}$, so that we could build another solution on

$$p^{-1}+q^{-1}+r^{-1} = 0$$

* Cf. **2**, 44. The form given there has to be diminished everywhere by 52 to reduce it to our form in which $x_r = -y_r$.

in place of (32). This solution is formally but not essentially different from (33), since we can clearly derive it by writing r^{-1}, p^{-1}, q^{-1} in place of p, q, r. Other formally-different solutions are, of course, obtainable by interchanging two of p, q, r.

10. The square of order eight

We derive a symmetric solution for the square of order eight, depending on one absolute parameter, from the symmetric square of order five in the following way. If s_r, p_r denote respectively the sums of powers and the sums of products of order r of five arguments $x_1,\ldots,x_5$, we have the elementary relation

$$s_6-p_1s_5+p_2s_4-p_3s_3+p_4s_2-p_5s_1=0.$$

Hence, if $p_1(=s_1)=0$, two such systems (x), (y) which have s_1, s_2, s_3, s_4, p_1, p_2, p_3, p_4 the same will also have s_6 the same. Hence, if $(x_1,\ldots,x_5;\ y_1,\ldots,y_5)$ is a solution of Tarry's equations (3) of order five in which

$$x_1+x_2+x_3+x_4+x_5=0, \tag{36}$$

we have

$$\sum_{r=1}^{5}x_r^k=\sum_{r=1}^{5}y_r^k \quad (k=2,4,6).$$

If, in addition,

$$x_5+y_5=0, \tag{37}$$

these equations reduce to

$$\sum_{r=1}^{4}x_r^k=\sum_{r=1}^{4}y_r^k \quad (k=2,4,6),$$

a solution of the symmetric equations (7) of order eight.

Since Tarry's equations (3) are unaltered by adding h to every argument, and since any pair of arguments x_α, y_β can be taken as x_5, y_5 in (37), we satisfy these two conditions (36), (37) if we can find h such that simultaneously

$$5h+\sum_{r=1}^{5}x_r=0, \qquad 2h+x_\alpha+y_\beta=0,$$

i.e. if

$$x_\alpha+y_\beta=\tfrac{2}{5}\sum x_r \tag{38}$$

for some α, β.

Turning to the asymmetric square of order five and its solution (25), (26), we have

$$\sum x_r=x_3+y_3=\{(r-s)p-(r+s)q\}\{(2r+s)p+qr\},$$
$$x_2+y_2=\{(r-s)p-(r+s)q\}\{(r-s)p+2qr\}.$$

Writing then

$$x_2+y_2=\tfrac{2}{5}(x_3+y_3)$$

we reject

$$(r-s)p-(r+s)q=0,$$

which gives a trivial solution, and take, sufficiently,

$$p = 8r, \qquad q = -(r-7s).$$

Removing the common factor $3r(r+s)$, subtracting $9r^2-14rs-7s^2$ from each argument, cancelling x_2, y_2 which are now numerically equal, and writing (for brevity) λ in place of r/s, we have, in effect, the solution

$$\left.\begin{aligned} \pm x &= 6\lambda^2+10\lambda-28, \quad 12\lambda^2-18\lambda-14, \\ &\qquad\qquad 15\lambda^2-28\lambda+21, \quad 9\lambda^2-32\lambda+7 \\ \pm y &= 12\lambda^2-38\lambda+14, \quad 15\lambda^2-24\lambda-7, \\ &\qquad\qquad 6\lambda^2-14\lambda+28, \quad 9\lambda^2+4\lambda-21 \end{aligned}\right\}. \tag{39}$$

Putting $\lambda = 2$ we have Tarry's solution*

$$\pm 16, \pm 2, \pm 25, \pm 21; \ \pm 14, \pm 5, \pm 24, \pm 23.$$

The corresponding square is centrally skew, and it is therefore sufficient to give only the first four columns, namely

$-6(\lambda-1)(\lambda-7)$	$-(\lambda-3)(9\lambda-7)$	$8(3\lambda-7)$	$-(3\lambda^2-6\lambda+7)$
$4(5\lambda-7)$	$-(3\lambda^2-6\lambda+7)$	$2(\lambda-3)(3\lambda+7)$	$(\lambda-7)(3\lambda-1)$
$(\lambda+1)(3\lambda+7)$	$-4(\lambda-7)$	$9\lambda^2-14\lambda-7$	$2(\lambda-3)(3\lambda-7)$
$-(3\lambda^2-6\lambda+7)$	$-2(\lambda-1)(3\lambda+7)$	$3(\lambda+1)(\lambda-7)$	$-4(9\lambda-7)$
$-7(\lambda-3)(3\lambda-1)$	$-8\lambda(3\lambda-7)$	$-(3\lambda-5)(5\lambda-7)$	$-2(9\lambda^2-14\lambda-7)$
$-(3\lambda-5)(9\lambda-7)$	$-2(3\lambda-1)(5\lambda-7)$	$-7(3\lambda^2-6\lambda+7)$	$-24\lambda(\lambda-1)$
$-8\lambda(3\lambda-7)$	$-3(9\lambda^2-14\lambda-7)$	$-2(\lambda-1)(9\lambda-7)$	$-7(\lambda+1)(3\lambda-5)$
$-2(9\lambda^2-14\lambda-7)$	$-7(\lambda+1)(3\lambda-5)$	$-4\lambda(3\lambda-1)$	$-(3\lambda+7)(5\lambda-7)$

(40)

Another solution, apparently distinct from (39), is

$$\left.\begin{aligned} \pm x &= 21\lambda^2+27\lambda-12, \quad 14\lambda^2-29\lambda-16, \\ &\qquad\qquad 35\lambda^2+7\lambda+8, \quad 28\lambda^2-\lambda-20 \\ \pm y &= 21\lambda^2-27\lambda-12, \quad 28\lambda^2+29\lambda-8, \\ &\qquad\qquad 14\lambda^2+7\lambda+20, \quad 35\lambda^2+\lambda-16 \end{aligned}\right\}, \tag{41}$$

and other such solutions are obtainable.

* Compare **2**, 44, where we diminish each number by 26 to obtain the symmetrical form given here.

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SOME PROPERTIES OF INTEGRAL FUNCTIONS OF FINITE ORDER

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[Received 29 December 1936; in revised form 2 February 1937]

1. Let $z = re^{i\theta}$. Consider the region $\Lambda(\alpha)$ of the plane defined by

$$|\theta| \leqslant \alpha, \qquad 0 < \alpha \leqslant \pi, \tag{1}$$

where α is fixed in any particular discussion. Let $f(z)$ be a function regular in the closed region $\Lambda(\alpha)$ except maybe at $z = 0$ where it is continuous as $z \to 0$ remaining in $\Lambda(\alpha)$. Let

$$M\{r, f, \Lambda(\alpha)\} = \max|f(z)|$$

as z varies in the sector common to $\Lambda(\alpha)$ and the circle $|z| \leqslant r$. The order ρ, the type $k\{f, \Lambda(\alpha)\}$, and the Lindelöf's function $h(\theta, f)$ are defined by the relations

$$\left.\begin{aligned} \rho &= \overline{\lim_{r\to\infty}} \frac{\log\log M\{r, f, \Lambda(\alpha)\}}{\log r} \\ k\{f, \Lambda(\alpha)\} &= \overline{\lim_{r\to\infty}} \frac{\log M\{r, f, \Lambda(\alpha)\}}{r^\rho} \\ h(\theta, f) &= \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r^\rho} \quad (|\theta| \leqslant \alpha) \end{aligned}\right\}. \tag{2}$$

When $f(z)$ is an integral function we write $k(f) = k\{f, \Lambda(\pi)\}$. Throughout this paper it is supposed that $0 < \rho < \infty$ and $k\{f, \Lambda(\alpha)\}$ is finite.*

1.1. Let $\{\lambda_n\}$ be a sequence such that

$$0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty$$

as $n \to \infty$. Let $n(r)$ denote the number of these λ_n's not exceeding r. If

$$\lim_{n\to\infty} \frac{n(r)}{r^\rho} = D,$$

we say that $\{\lambda_n\}$ is measurable of order ρ and density D. When $\rho = 1$, the index of condensation† δ of $\{\lambda_n\}$ is given by

$$\delta = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log \frac{1}{|\sigma'(\lambda_n)|} \tag{3}$$

* All the results of this paper hold if, instead of ρ, we work with any Lindelöf proximate order $\rho(r)$. For simplicity we have confined ourselves to the case $\rho(r) \equiv \rho$.

† The index of condensation of $\{\lambda_n\}$ can be defined directly and shown to be

where $$\sigma(z) = \prod_{n=1}^{\infty}\prod \left(1-\frac{z^2}{\lambda_n^2}\right). \tag{4}$$

When $\rho \neq 1$, we denote by $\delta(\rho)$ the index of condensation of $\{\lambda_n^\rho\}$ which is measurable of order one and density D. We always suppose $D > 0$.

1.2. The following result has been proved by Bernstein:*

THEOREM A. *Let $\{\lambda_n\}$ be a measurable sequence of order one and density D; let its index of condensation be zero. Let $f(z)$ be regular, of order one, and finite type in $\Lambda(\frac{1}{2}\pi)$. Let $h(\frac{1}{2}\pi, f)+h(-\frac{1}{2}\pi, f) < 2\pi D$. Then*

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n}\log|f(\lambda_n)| = h(0,f) = \overline{\lim_{r\to\infty}}\frac{\log|f(r)|}{r}.$$

1.3. The proof given by Bernstein for Theorem A depends on the general properties of Dirichlet's series. This paper contains a general result of which Theorem A is a special case [Theorem 2 below with $\rho = 1$, $\delta(\rho) = \delta = 0$]. The proof is direct and simple and is based on an approximate interpolation-formula established in Theorem 1. The main result is contained in Theorem 2, from which several properties of integral functions of order $\rho \geqslant \frac{1}{2}$ are derived. This method does not seem to apply to the case $0 < \rho < \frac{1}{2}$ which is treated by a different method.

1.4. The following two lemmas give the requisite properties of the function $h(\theta, f)$, and $\sigma(z)$ given by (4) which are used in the sequel.†

LEMMA 1. *Let $f(z)$ be of order ρ and finite type in $\Lambda(\alpha)$ $(\alpha \leqslant \pi/2\rho)$. Then the function $h(\theta, f)$ has the following properties:*

(i) *if, for a single θ_0 in $|\theta| < \alpha$, $h(\theta_0, f) = -\infty$, then $h(\theta, f) = -\infty$ for all θ in $|\theta| < \alpha$ and*

$$\lim_{r\to\infty}\frac{1}{r^\rho}\log|f(re^{i\theta})| = -\infty, \tag{5}$$

uniformly in $|\theta| \leqslant \alpha-\eta$ $(0 < \eta < \alpha)$;

(ii) *except in the case mentioned in* (i), *$h(\theta, f)$ is continuous in*

given by (3) when $\{\lambda_n\}$ is measurable. (See V. Bernstein, 'Séries de Dirichlet', *Borel Tracts* (1933), 22–7.) The index δ is non-negative and is always zero when $\lim_{n\to\infty}(\lambda_n-\lambda_{n-1}) > 0$.

* Bernstein, l.c., 235, Theorem 2.

† Ibid. 300–1, 279–81. The properties of $h(\theta, f)$ when $\rho = 1$ are treated here. The general case can be reduced to this by a transformation of the form $z^\rho = Z$.

$|\theta| < \alpha$; *moreover, if* $H(\theta) = A\cos\theta\rho + B\sin\theta\rho$ *where* $H(\theta_1) = h(\theta_1, f)$, $H(\theta_2) = h(\theta_2, f)$, $\sigma \leqslant \theta_1 - \theta_2 < \pi/\rho$, $|\theta_1| \leqslant \alpha$, $|\theta_2| \leqslant \alpha$, *then*

$$\left.\begin{aligned} h(\theta, f) &\leqslant H(\theta) \quad (\theta_2 \leqslant \theta \leqslant \theta_1) \\ h(\theta, f) &\geqslant H(\theta) \quad (|\theta| \leqslant \alpha;\ \theta \text{ not in } \theta_2 < \theta < \theta_1) \end{aligned}\right\}; \tag{6}$$

(iii) *if* $\alpha = \pi/2\rho$ *and* $h(\pi/2\rho, f) + h(-\pi/2\rho, f) < 0$, *then* $f(z) \equiv 0$.

LEMMA 2. *The function* $\sigma(z)$ *defined by* (4) *satisfies the relation*

$$\lim_{r\to\infty} \frac{\log|\sigma(re^{i\theta})|}{r} = \pi D|\sin\theta| \tag{7}$$

uniformly in any angle not containing $\theta = 0$ *or* π; *given a positive* ϵ *we can find a* $\gamma = \gamma(\epsilon)$ *such that*

$$|\sigma(z)| \leqslant e^{\epsilon|z|}, \tag{8}$$

where $|\theta| \leqslant \gamma$ *or* $|\pi - \theta| \leqslant \gamma$.

2. We first establish the following approximate interpolation formula:

THEOREM 1. *Let* $f(z)$ *be regular, of order one, and finite type in* $\Lambda(\alpha)$ $(\alpha \leqslant \frac{1}{2}\pi)$. *Let* $\{\lambda_n\}$ *be a measurable sequence of order one, density* D, *and finite index of condensation* δ. *Let*

(i) $h(\alpha, f) < \pi D\sin\alpha$, $\quad h(-\alpha, f) < \pi D\sin\alpha$;

(ii) $\overline{\lim_{n\to\infty}} \dfrac{1}{\lambda_n}\log|f(\lambda_n)| = -d \quad (d > \delta)$;

(iii) $H(z) = \dfrac{f(z)}{\sigma(z)} - \displaystyle\sum_{n=1}^{\infty} \frac{f(\lambda_n)}{\sigma'(\lambda_n)} \frac{1}{z-\lambda_n}$.

Then $\qquad H(z) \to 0$ *uniformly as* $|z| \to \infty$ *in* $\Lambda(\alpha)$. $\qquad (9)$

Proof. Firstly, we note that

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n}\log\left|\frac{f(\lambda_n)}{\sigma'(\lambda_n)}\right| \leqslant -d + \delta < 0, \tag{10}$$

so that the series on the right-hand side of (iii) converges absolutely and uniformly except at the points $z = \lambda_n$. It follows that $H(z)$ is regular in $\Lambda(\alpha)$. By using the method employed elsewhere* it can be shown that $H(z)$ is of order one at most in $\Lambda(\alpha)$. First, let $\alpha < \frac{1}{2}\pi$. On $\theta = \pm\alpha$, by (7), (10), $|zH(z)|$ is bounded as $|z| \to \infty$, since $h(\alpha, f) < \pi D\sin\alpha$ and $h(-\alpha, f) < \pi D\sin\alpha$. Hence, since $H(z)$ is

* *J. Indian Math. Soc.* (New Series), 2 (1936), 4–6. The method employed here should be supplemented by reasoning of the type used to prove theorems of Phragmén-Lindelöf type after splitting up the angle $|\theta| \leqslant \alpha$ into two convenient parts. The details are not difficult though somewhat cumbersome; hence they are omitted in the body of this paper.

of order one at most in $\Lambda(\alpha)$ and $2\alpha < \pi$, we conclude by a well-known theorem of Phragmén and Lindelöf,* that $|zH(z)|$ is bounded uniformly in $\Lambda(\alpha)$ as $|z| \to \infty$. Hence $H(z)$ satisfies (9) in $\Lambda(\alpha)$. Next, let $\alpha = \frac{1}{2}\pi$. Here,† in virtue of Lemma 1 (ii), we can choose η $(0 < \eta < \frac{1}{2}\pi)$, so that

$$h(\tfrac{1}{2}\pi-\eta, f) < \pi D \sin(\tfrac{1}{2}\pi-\eta), \qquad h(-\tfrac{1}{2}\pi+\eta, f) < \pi D \sin(\tfrac{1}{2}\pi-\eta),$$

By what is already proved $H(z)$ satisfies (9) in $|\theta| \leqslant \frac{1}{2}\pi-\eta$. On the sides of the angles $\frac{1}{2}\pi-\eta \leqslant \theta \leqslant \frac{1}{2}\pi$, $-\frac{1}{2}\pi \leqslant \theta \leqslant -\frac{1}{2}\pi+\eta$, we find, as before, that $|zH(z)|$ is bounded as $|z| \to \infty$ and another application of the Phragmén-Lindelöf theorem shows that (9) holds in these angles as well. So the theorem is proved.

2.1. We can now prove the chief result of this paper. We suppose $\rho \geqslant \frac{1}{2}$.

THEOREM 2. *Let $f(z)$ be of order ρ and finite type in $\Lambda(\pi/2\rho)$. Let $\{\lambda_n\}$ be a measurable sequence of order ρ and density D. Let $\delta(\rho)$, the index of condensation of $\{\lambda_n^\rho\}$, be finite. Let*

(i) $h\left(\frac{\pi}{2\rho}, f\right)+h\left(-\frac{\pi}{2\rho}, f\right) < 2\pi D,$

(ii) $\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^\rho} \log|f(\lambda_n)| = d.$

Then $\quad d \leqslant h(0, f) \leqslant d+\delta(\rho).$

Proof. We can suppose $\rho = 1$ since the general case can be reduced to this by a transformation of the form $z^\rho = Z$. Therefore let $\rho = 1$. We can suppose without loss of generality that $h(\frac{1}{2}\pi, f) = h(-\frac{1}{2}\pi, f)$, since, otherwise, we can deal with the function

$$g(z) = f(z)\exp\left[\frac{z}{2i}(h(-\tfrac{1}{2}\pi, f)-h(\tfrac{1}{2}\pi, f))\right],$$

for which $h(\frac{1}{2}\pi, g) = h(-\frac{1}{2}\pi, g) = \frac{1}{2}[h(\frac{1}{2}\pi, f)+h(-\frac{1}{2}\pi, f)]$ while

$$h(0, g) = h(0, f).$$

It is evident that $d \leqslant h(0, f)$. Let β be any number greater‡ than $d+\delta$ where $\delta = \delta(1)$. Let $F(z) = e^{-\beta z} f(z)$. Then

$$h(\tfrac{1}{2}\pi, F) = h(\tfrac{1}{2}\pi, f) < \pi D \quad \text{and} \quad h(-\tfrac{1}{2}\pi, F) = h(-\tfrac{1}{2}\pi, f) < \pi D$$

* G. Valiron, *Lectures on Integral Functions*, p. 125.

† If for some θ_0, $|\theta_0| < \frac{1}{2}\pi$, $h(\theta_0, f) = -\infty$, then $h(\theta, f) = -\infty$ for all θ in $|\theta| < \frac{1}{2}\pi$, by Lemma 1, (i). Hence in this case the choice of η as desired can always be made.

‡ When $d = -\infty$, β can be taken to be any large negative number.

while $$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n}\log|F(\lambda_n)| = -\beta+d.$$

But $\beta-d > \delta$. Hence by Theorem 1 (with $\alpha = \frac{1}{2}\pi$)

$$\left|\frac{F(z)}{\sigma(z)} - \sum_{n=1}^{\infty} \frac{F(\lambda_n)}{\sigma'(\lambda_n)}\frac{1}{z-\lambda_n}\right| \to 0, \tag{11}$$

as $|z| \to \infty$ uniformly in $\Lambda(\frac{1}{2}\pi)$. Let $C(p)$ denote the system of circles with centres at $z = \lambda_n$ and radii λ_n^{-p} $(p > 1)$ so that $\sum \lambda_n^{-p}$ converges. Then, for z in $\Lambda(\frac{1}{2}\pi)$ but outside the circles $C(p)$,

$$\left|\sum_{n=1}^{\infty} \frac{F(\lambda_n)}{\sigma'(\lambda_n)}\frac{1}{z-\lambda_n}\right| \leqslant A = A(p), \tag{12}$$

where $A(p)$ is a constant depending on p. This is because $\beta-d > \delta$. Hence, given a positive ϵ, we conclude by (8), (11), (12), that

$$|F(z)| \leqslant Pe^{\epsilon|z|}, \tag{13}$$

where $|\theta| \leqslant \gamma = \gamma(\epsilon)$ and $P = P(p, \epsilon)$ is a constant depending on p and ϵ while z lies in $|\theta| \leqslant \gamma$ but outside the circles $C(p)$. If ϵ be so small that $1-2\cos\gamma < 0$, we get from (13),

$$|e^{-2\epsilon z}F(z)| \leqslant P \tag{14}$$

for z in $|\theta| \leqslant \gamma$ but outside $C(p)$. Since $e^{-2\epsilon z}F(z)$ is regular in $|\theta| \leqslant \gamma$ and $\sum \lambda_n^{-p}$ converges, it follows by the maximum-modulus principle that (14) holds throughout $|\theta| \leqslant \gamma$. Hence $h(0, F) \leqslant 2\epsilon$, so that $h(0, f) \leqslant \beta+2\epsilon$. Since ϵ is as small as we please and β is *any* number greater than $d+\delta$, we get $h(0, f) \leqslant d+\delta$ which proves the theorem.*

3. The following two theorems hold for integral functions of any finite order $\rho \geqslant \frac{1}{2}$.

THEOREM 3. *Let $f(z)$ be an integral function of order ρ and finite type. Let $\{\lambda_n\}$ be a measurable sequence of order ρ and density D. Let $\delta(\rho)$, the index of condensation of $\{\lambda_n^\rho\}$, be finite. Let*

(i) $h(\theta, f) < \pi D \qquad \left(\frac{\pi}{2\rho} \leqslant \theta \leqslant 2\pi - \frac{\pi}{2\rho}\right)$,

* I am indebted to a referee for pointing out that Theorem 2 may not hold for $\Lambda(\alpha)$ $(\alpha < \pi/2\rho)$ when the boundary condition (i) is replaced by

$$h(\alpha, f)+h(-\alpha, f) < 2\pi D\sin\alpha\rho.$$

That this may be the case is seen by considering $\exp(-\beta z)\sin\pi z$, $\beta > 0$, $\lambda_n = n$, $\alpha < \frac{1}{2}\pi$, and $\rho = 1$. But under the above condition it can be proved that $h(0, f) \leqslant \max\{0, d+\delta(\rho)\}$.

(ii) $\overline{\lim}_{n\to\infty} \frac{1}{\lambda_n^\rho} \log|f(\lambda_n)| = -d$, *where* $d \geqslant \pi D + \delta(\rho)$.

Then $f(z) \equiv 0$.

Proof. Since $h(\pi/2\rho, f) < \pi D$, $h(-\pi/2\rho, f) < \pi D$, Theorem 2 gives $h(0, f) \leqslant -d + \delta(\rho) \leqslant -\pi D$. Using this and Lemma 1 (i) or (ii), it is easily seen that $h(\theta, f) < \pi D$ for all θ in $0 \leqslant \theta \leqslant 2\pi$. Let

$$h\left(\frac{\pi}{\rho}, f\right) < \beta < \pi D \quad \text{and} \quad F(z) = e^{\beta z^\rho} f(z).$$

Then

$$h(0, F) \leqslant \beta - \pi D < 0 \quad \text{and} \quad h\left(\frac{\pi}{\rho}, F\right) = -\beta + h\left(\frac{\pi}{\rho}, f\right) < 0.$$

Hence $F(z) \equiv 0$ by Lemma 1 (iii). Therefore also $f(z) \equiv 0$.

3.1. THEOREM 4. *Let* $f(z)$, $\{\lambda_n\}$ *be as in Theorem* 3. *Let*

$$\overline{\lim}_{n\to\infty} \frac{1}{\lambda_n^\rho} \log|f(\lambda_n)| = -d.$$

Then $k(f) \geqslant \min\{d - \delta(\rho), \pi D\}$ *or else* $f(z) \equiv 0$.

Proof. Let $\eta = \min\{d - \delta(\rho), \pi D\}$. If $\eta \leqslant 0$, the result to be proved is trivial. Let $\eta > 0$ and $k(f) < \eta$. Then

$$h\left(\frac{\pi}{2\rho}, f\right) + h\left(-\frac{\pi}{2\rho}, f\right) \leqslant 2k(f) < 2\eta < 2\pi D$$

so that, by Theorem 2,

$$h(0, f) \leqslant -d + \delta(\rho) \leqslant -\eta.$$

Let $$k(f) < \beta < \eta \quad \text{and} \quad F(z) = e^{\beta z^\rho} f(z).$$

Then $h(0, F) \leqslant \beta - \eta < 0$ and $h\left(\frac{\pi}{\rho}, F\right) \leqslant -\beta + k(f) < 0$.

Hence $F(z) \equiv 0$ and therefore $f(z) \equiv 0$, by Lemma 1 (iii).

4. A special case of interest is when $\delta(\rho) = 0$ which always happens when $\overline{\lim}_{n\to\infty} (\lambda_n^\rho - \lambda_{n-1}^\rho) > 0$. Theorem 4 gives as a special case

THEOREM 5. *Let* $f(z)$ *be an integral function of order* $\rho \geqslant \frac{1}{2}$ *and finite type. Let* $\{\lambda_n\}$ *be a measurable sequence of order* ρ *and density* D *while* $\overline{\lim}_{n\to\infty} (\lambda_n^\rho - \lambda_{n-1}^\rho) > 0$. *Let*

$$\overline{\lim}_{n\to\infty} \frac{1}{\lambda_n^\rho} \log|f(\lambda_n)| = -d.$$

Then $k(f) \geqslant \min(d, \pi D)$ *or else* $f(z) \equiv 0$.

4.1. *Remarks.* When $\rho = 1$, Theorem 4 becomes

THEOREM 6. *Let $f(z)$ be an integral function of order one and finite type, $\{\lambda_n\}$ a measurable sequence of order one, density D, and index of condensation δ. Let*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log|f(\lambda_n)| = -d.$$

Then $k(f) \geqslant \min(d-\delta, \pi D)$ *or else* $f(z) \equiv 0$.

In an earlier note I have proved* that under the hypothesis of Theorem 6, $k(f) \geqslant \min(d-2\delta, \pi D)$. The result of Theorem 6 is sharper, for $d-\delta \geqslant d-2\delta$ since $\delta \geqslant 0$. When $\delta = 0$ the difference between the two results does not appear explicitly.

4.2. If the hypothesis that $f(z)$ is an integral function be dropped, some boundary condition is necessary to get significant results. We state the following for $\rho = 1$.

THEOREM 7. *Let $f(z)$ be regular, of order one, and finite type in $\Lambda(\frac{1}{2}\pi)$. Let the following conditions hold:*

(i) $h(\frac{1}{2}\pi, f)+h(-\frac{1}{2}\pi, f) < 2\pi D$,

(ii) $\displaystyle\int^{\infty} \frac{\log^+|f(iy)|+\log^+|f(-iy)|}{y^2}\, dy$ *be convergent,*

(iii) $\overline{\lim_{n\to\infty}} \dfrac{1}{\lambda_n} \log|f(\lambda_n)| = -\infty$,

where $\{\lambda_n\}$ is a measurable sequence of order one, density D, and finite index of condensation. Then $f(z) \equiv 0$. In particular, (i) *and* (ii) *hold when $f(\pm iy) = O(1)$.*

Proof. By (i) and Theorem 2 we get $h(0, f) = -\infty$, so that by Lemma 1 (i)

$$\lim_{r\to\infty} \frac{1}{r} \log|f(re^{i\theta})| = -\infty, \tag{15}$$

uniformly for $|\theta| \leqslant \frac{1}{2}\pi-\eta$, $0 < \eta < \frac{1}{2}\pi$. By using the properties of Green's function† for a semicircle, we get

$$\log|f(x)| \leqslant 2x \overline{\lim_{r\to\infty}} \int_{-r}^{r} \frac{\log^+|f(iy)|}{x^2+y^2}\, dy + \overline{\lim_{r\to\infty}} \frac{4x}{r} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log|f(re^{i\theta})| \cos\theta\, d\theta, \tag{16}$$

* See above, pp. 103–6.

† Cf. R. and F. Nevanlinna, *Acta. Sci. Fennicae*, 50, No. 5, 1–46.

where $x > 0$. By (15), we find that

$$\lim_{r\to\infty} \frac{4x}{r} \int_{\eta-\frac{1}{2}\pi}^{\eta+\frac{1}{2}\pi} \log|f(re^{i\theta})|\cos\theta\, d\theta = -\infty,$$

for $0 < \eta < \frac{1}{2}\pi$, while, by hypothesis, the remaining expressions in (16) remains less than a finite constant depending on x and η. Hence $\log|f(x)| \leqslant -\infty$ or $f(x) \equiv 0$ for $x > 0$. Hence $f(z) \equiv 0$.

5. We now treat the case $0 < \rho < \frac{1}{2}$. Since, here, $\pi/2\rho > \pi$, Theorem 2 and hence the reasoning employed to derive Theorems 3 and 4 do not apply. But we can employ another method* I have used in several other places and obtain a result analogous to Theorem 4.

THEOREM 8. *Let $f(z)$ be an integral function of order ρ $(0 < \rho < \frac{1}{2})$ and finite type. Let $\{\lambda_n\}$ be measurable of order ρ, and density D. Let*

$$\sigma_\rho(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)$$

and

$$\delta_1(\rho) = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^\rho} \log\frac{1}{|\sigma_\rho'(\lambda_n)|}.$$

Let

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^\rho} \log|f(\lambda_n)| = -d < 0.$$

Then $k(f) \geqslant \min\{d-\delta_1(\rho),\ \pi D \operatorname{cosec}\pi\rho\}$ *or else* $f(z) \equiv 0$.

Proof. Let $\eta = \min\{d-\delta_1(\rho), \pi D \operatorname{cosec}\pi\rho\}$. If $\eta \leqslant 0$, the result to be proved is trivial. Let $\eta > 0$ and $k(f) < \eta$. Let

$$F_m(z) = z^m f(z) \quad (m = 0, 1, 2, \ldots).$$

Then

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^\rho} \log\left|\frac{F_m(\lambda_n)}{\sigma_\rho'(\lambda_n)}\right| \leqslant -d+\delta_1(\rho) \leqslant -\eta < 0,$$

so that the series on the right of

$$H_m(z) = \frac{F_m(z)}{\sigma_\rho(z)} - \sum_{n=1}^{\infty} \frac{F_m(\lambda_n)}{\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n} \tag{17}$$

converges absolutely and uniformly except at the points $z = \lambda_n$. As in the proof† of Theorem 1, we can conclude that $H_m(z)$ is an integral function of order ρ at most. Since‡

$$\log\sigma_\rho(-r) \sim (\pi D \operatorname{cosec}\pi\rho) r^\rho$$

* Cf. *J. of London Math. Soc.* 11 (1936), 247–50; *J. Indian Math. Soc.* (New Series), 2 (1936), 131–40.

† Cf. V. G. Iyer, *J. Indian Math. Soc.* (New Series), 2 (1936), 57–9.

‡ Paley and Wiener, *Fourier Transforms in the Complex Domain*, p. 79.

as $r \to \infty$, we find from (17) that

$$H_m(-r) \to 0$$

as $r \to \infty$ since $k(F_m) < \eta \leqslant \pi D \operatorname{cosec} \pi\rho$ and $\eta > 0$. Hence by an application of the Phragmén-Lindelöf theorem we conclude that $H_m(z) \equiv 0$, so that

$$z^m \frac{f(z)}{\sigma_\rho(z)} = \sum_{n=1}^{\infty} \frac{\lambda_n^m f(\lambda_n)}{\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n}. \tag{18}$$

Let $g(z) = c_0+c_1z+c_2z^2+ \ldots$ be any integral function of order ρ and type less than η, so that

$$\sum_0^{\infty} |c_m||z|^m \leqslant e^{\mu|z|^\rho} \quad (\mu < \eta).$$

Then the double series

$$\sum_{(n,m)} \left| \frac{c_m \lambda_n^m f(\lambda_n)}{\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n} \right|$$

converges except at the points $z = \lambda_n$. Therefore, (18) gives

$$\frac{f(z)g(z)}{\sigma_\rho(z)} = \sum_{n=1}^{\infty} \frac{f(\lambda_n)g(\lambda_n)}{\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n}. \tag{19}$$

In (19) we can take $g(z) = z^m f(z)$ $(m = 0, 1, 2, \ldots)$ so that we get

$$\frac{z^m[f(z)]^2}{\sigma_\rho(z)} = \sum_{n=1}^{\infty} \frac{\lambda_n^m[f(\lambda_n)]^2}{\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n}. \tag{20}$$

Starting from (20) we can repeat the process since $d > 0$ and so $-2d+\delta_1(\rho) < -\eta$, and prove successively that*

$$\frac{[f(z)]^p}{\sigma_\rho(z)} = \sum_{n=1}^{\infty} \frac{[f(\lambda_n)]^p}{\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n}, \tag{21}$$

where $p = 1, 2, 3, \ldots$. Now it is easy to see that the double series

$$\sum_{\substack{(n,p) \\ p \geqslant 1}} \left| \frac{[f(\lambda_n)]^p}{(p-1)!\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n} \right|$$

converges so that (21) gives

$$\frac{f(z)e^{f(z)}}{\sigma_\rho(z)} = \sum_{n=1}^{\infty} \frac{f(\lambda_n)e^{f(\lambda_n)}}{\sigma_\rho'(\lambda_n)} \frac{1}{z-\lambda_n}. \tag{22}$$

from which we conclude,† since $f(\lambda_n) = O(1)$, $e^{f(\lambda_n)} = O(1)$, that

* We cannot assert that (21) is true for $p = 0$, since $\delta_1(\rho)$ may be positive or zero.

† Cf. the reference to the *J. Indian Math. Soc.* 2 (1936), 57–9.

$f(z)e^{f(z)}$ is of order ρ at most, which can happen only if $f(z)$ is a polynomial and, since $f(\lambda_n) \to 0$ as $\lambda_n \to \infty$, it follows that $f(z) \equiv 0$.

5.1. *Remarks.* The quantities $\delta_1(\rho)$ as defined above and the index $\delta(\rho)$ of $\{\lambda_n^\rho\}$ are connected by the relation*

$$\delta(\rho) = \delta_1(\rho) + \pi D \cot \pi\rho.$$

Since $0 < \rho < \frac{1}{2}$, $\pi D \cot \pi\rho > 0$ and therefore $d - \delta_1(\rho) > d - \delta(\rho)$ while $\pi D \operatorname{cosec} \pi\rho > \pi D$. Hence

$$\min\{d - \delta_1(\rho), \pi D \operatorname{cosec} \pi\rho\} > \min\{d - \delta(\rho), \pi D\},$$

so that when $0 < \rho < \frac{1}{2}$, Theorem 8 gives more precise information than the analogue of Theorem 4.

5.2. The following special case of Theorems 4 and 8 is worthy of notice:

THEOREM 9. *Let $f(z)$ be an integral function of positive order ρ and minimal type: that is, $k(f) = 0$. Let $\{\lambda_n\}$ be a measurable sequence of order ρ and positive density D however small. Let $\delta(\rho) = 0$. Then*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^\rho} \log |f(\lambda_n)| = k(f) = 0.$$

Proof. Let

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^\rho} \log |f(\lambda_n)| = -d.$$

Then $-d \leqslant k(f) = 0$: that is $d \geqslant 0$. If possible, let $d > 0$. Then $k(f) = 0 < d$ and $k(f) = 0 < \pi D$, so that $k(f) < \min(d, \pi D)$ which shows that, when $\rho \geqslant \frac{1}{2}$, $f(z) \equiv 0$ by Theorem 4. But in that case $k(f) = -\infty$, which contradicts the hypothesis $k(f) = 0$. Hence $d = 0$ when $\rho \geqslant \frac{1}{2}$. If $0 < \rho < \frac{1}{2}$, we get, similarly, if $d > 0$,

$$k(f) = 0 < \min\{d - \delta_1(\rho), \pi D \operatorname{cosec} \pi\rho\}$$

so that Theorem 8 gives a contradiction.

5.3. Theorem 9 is closely allied to Faber's result that, for an integral function of order one and minimal type $|f(n)| \geqslant e^{-\alpha n}$ for any positive α and for almost all positive integers. The proof given by J. M. Whittaker† for the above result can be modified to show that $|f(n^{1/\rho})| \geqslant e^{-\alpha n}$ for any positive α and for almost all integers when $f(z)$ is of order ρ and minimal type. Theorem 9 is a partial

* 'Some theorems on integral functions bounded at a sequence of points', communicated to the London Math. Soc.

† *Interpolatory Function Theory* (Cambridge Tract 33), 57–9.

generalization when $\{\lambda_n\}$ replaces $\{n^{1/\rho}\}$. It seems doubtful whether Theorem 9 can hold under the sole assumptions

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r^\rho} > 0 \quad \text{and} \quad \delta(\rho) = 0.$$

5.4. *Conclusion.* Let $f(z)$ be of order one and finite type in

$$\Lambda(\alpha) \quad (\alpha \leqslant \tfrac{1}{2}\pi).$$

Let $\{\lambda_n\}$ be measurable of order one and density D; let its index of condensation be zero. Then, using Lemma 1 (ii), it is possible to find a constant b such that

$$h(\theta, f) \leqslant h(0, f)\cos\theta + b|\sin\theta| \quad (|\theta| \leqslant \alpha). \tag{23}$$

Bernstein derives Theorem A from the more general result that* *if*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n}\log|f(\lambda_n)| = d < h(0,f),$$

the least value of b in (23) *cannot be less than πD.* It may be noted that this result also can be proved by using Theorem 1 on the same lines as the proof of Theorem 2. Putting $F(z) = f(z)e^{-h(0,f)z}$ we have to show that if $h(\theta, F) < \pi D|\sin\theta|$ $(|\theta| \leqslant \alpha)$, then

$$h(0, F) = 0 = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n}\log|f(\lambda_n)|,$$

and this can be easily done by applying Theorem I to the function $e^{-\beta z}F(z)$, $\beta > 0$ being arbitrarily small since $d \leqslant h(0, F) = 0$ and $\delta = 0$.

* Bernstein, l.c., Theorem I, p. 230.

A FUNDAMENTAL LEMMA ON NORMAL COORDINATES AND ITS APPLICATIONS

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(Received 16 March 1937)

SECTION 1 of this paper contains a fundamental lemma on normal coordinates which can be described briefly by saying that any point P of a general space of paths admits a neighbourhood which can be covered *uniformly* by normal coordinates. In Section 2 it is shown how the lemma can be applied to give a simple demonstration of Whitehead's convexity theorem, and in Section 3 a useful theorem concerning normal coordinates on a compact set is given. In each case it is found that the methods of proof have been both simplified and unified by the use of this lemma and the properties of normal coordinates.

1. A general space of paths* S is an n-dimensional coordinate manifold in which a set of paths is defined by the solutions of

$$x^{i\prime\prime} = H^i(x, x') \quad (i = 1,..., n). \tag{1}$$

$H^i(x, x')$ will be assumed to be positively homogeneous of the second degree in x', and of class C^2 except at the points $(x, 0)$, where it is of class C^1. It has been shown† that to each point x_0 of S there exists a neighbourhood $N(x_0)$ coverable by normal coordinates, and indeed an $N(x_0)$ with the property that there is one and only one path joining x_0 to any point x of $N(x_0)$ which lies entirely in $N(x_0)$. Such a domain of normal coordinates (y) is given by $\sum_{i=1}^{n} y^i y^i < \rho$ for sufficiently small ρ. Let the mapping of normal coordinates (y) on S be given by

$$x^i = \phi^i(x_0, y) \quad (i = 1,..., n), \tag{2.1}$$

where $\phi^i(x_0, 0) = x_0^i$ and the paths issuing from x_0 have the equations $x^i(t) = \phi^i(x_0, \xi t)$ $(0 \leqslant t; \xi \neq 0)$. The functions $\phi^i(x_0, y)$ are defined in a domain $|x_0^\alpha - \bar{x}_0^\alpha| < a$, $|y^\alpha| < b$. They are of class C^2 in this domain

* J. Douglas, *Annals of Math.* 29 (1927–8), 143–68.

† This was first shown by J. H. C. Whitehead in his paper, *Annals of Math.* 36 (1935), 680–704. See also T. Y. Thomas, *Proc. N. A. S.* 22 (1936), 309–12; also W. Mayer, *Lecture Notes on the Calculus of Variations*, Princeton, 1935–6, and a forthcoming paper by W. Mayer.

except at the points $(x_0, y = 0)$, and of class C^1 without restriction; the Jacobian $|\partial\phi/\partial y|$ is equal to unity at $(x_0, y = 0)$.

It is to be noted that two types of normal coordinates exist, one in which paths issuing from a point P have linear equations and one in which the paths terminating at a point P have linear equations. This distinction arises from the fact that, in general, paths are oriented, i.e. a path from a point P to a point Q need not be a path from Q to P. The lemma to be proved is valid for each type of normal coordinates.

To the equations (2.1) add the equations

$$x_0^i = x_0^i \quad (i = 1,\ldots, n). \tag{2.2}$$

The entire set (2) defines a continuous single-valued mapping of a neighbourhood of a point $(x_0, y = 0)$ in the product space* $\{x\}\times\{y\}$ on a neighbourhood of the point (x_0, x_0) in the product space* $\{x\}\times\{x\}$, where $(x_0, y = 0)$ maps into (x_0, x_0). All the functions in (2) are of class C^1, and the Jacobian, being equal to $|\partial\phi/\partial y|$, is equal to unity at $(x_0, y = 0)$; therefore there exists a neighbourhood $V(x_0, y = 0)$ and a neighbourhood $V(x_0, x_0)$ which are in one-to-one correspondence under (2). These neighbourhoods will be taken so small that the Jacobian does not vanish at any point of them.

Let $U_m(x_0)$ be a sequence of neighbourhoods of x_0 in the n-dimensional space S; let $U_m(x_0)$ contain $U_{m+1}(x_0)$ and let $U_m(x_0)$ shrink to the point x_0 as m approaches infinity. For all m greater than a certain N_1,

$$U_m(x_0)\times U_m(x_0) \subset V(x_0, x_0).$$

Let R_m $(m > N_1)$ be that map of $U_m(x_0)\times U_m(x_0)$ which lies in $V(x_0, y = 0)$. Let ρ_m be the least upper bound of $\sum_{i=1}^{n} y^i y^i$ for all y such that $(x, y) \in R_m$; ρ_m is obviously positive, and since R_m is open $\sum_{i=1}^{n} y^i y^i$ does not attain the value ρ_m for any (x, y) in R_m. Obviously ρ_m is not greater than the diameter d of R_m, d being measured by an artificial Euclidean metric in $V(x_0, y = 0)$ with the x_0^i and y^i taken as cartesian coordinates. Hence, from the continuity of the mapping of $U_m(x_0)\times U_m(x_0)$ on R_m, and the fact that the $U_m(x_0)$ shrink into the point x_0, it follows that ρ_m approaches zero as m approaches

* These are not *symmetric* product spaces, but consist of *ordered* pairs of points.

infinity. Let S_m be the set of all y such that $\sum_{i=1}^{n} y^i y^i < \rho_m$; R_m, then, is contained in $U_m(x_0) \times S_m$. For all m greater than a certain $N_2 \geqslant N_1$,

$$U_m(x_0) \times S_m \subset V(x_0, y = 0).$$

$U_m(x_0) \times S_m$, then, is in one-to-one correspondence with its map in $V(x_0, x_0)$. Because of the nature of this mapping (2), the set of points $(\bar{x}, y)$, for fixed $\bar{x}$ in $U_m(x_0)$ and for any y in S_m, is in one-to-one correspondence with a set of points $(\bar{x}, x)$ in $V(x_0, x_0)$ with the same fixed $\bar{x}$; thus for fixed $\bar{x}$ in $U_m(x_0)$ there is a one-to-one mapping of S_m on some neighbourhood of $\bar{x}$ in S. Hence a spherical domain of normal coordinates exists with origin at any point x of $U_m(x_0)$ and with fixed radius ρ_m. Since $U_m(x_0) \times S_m \supset R_m$, the map of $U_m(x_0) \times S_m$ contains $U_m(x_0) \times U_m(x_0)$; therefore this spherical domain of normal coordinates covers $U_m(x_0)$. We thus have the fundamental lemma:

To each point P of a general space of paths, S, there exists a neighbourhood $U(P)$ such that $U(P)$ may be covered by a domain of normal coordinates (y) having origin at any point of $U(P)$ and containing all y such that $\sum_{i=1}^{n} y^i y^i < \rho$ for a fixed positive ρ.

2. An application of this theorem will be made in the proof of

THEOREM 1. *In a general space of paths S there exists to each point P of S a simple convex neighbourhood $N(P)$.*

This interesting theorem was first proved by Whitehead* using a rather involved form of existence theorem on ordinary differential equations.

A neighbourhood will be called convex if there exists at least one path joining any two points of the neighbourhood and lying entirely in it; it will be called simple if there is only one such path.

Let $D(x_0)$ be a domain containing x_0 and lying in a fixed coordinate neighbourhood of x_0. Consider a family of hypersurfaces

$$V(x) = k \quad (k > 0),$$

such that the following properties are satisfied:

(i) $V(x)$ has continuous second partial derivatives;

(ii) each hypersurface of the family is a topological sphere containing x_0 in its interior;

* J. H. C. Whitehead, *Quart. J. of Math.* (Oxford), 3 (1932), 33–42; 4 (1933), 226–7.

(iii) one and only one hypersurface passes through each point of $D(x_0)$ with the exception of x_0;

(iv) the hypersurface given by k_1 contains that given by k_2 if k_1 is greater than k_2;

(v) the hypersurfaces shrink to the point x_0 as k approaches zero.

As an example of such a family take

$$V(x) \equiv \sum_{i=1}^{n} (x^i - x_0^i)^2 = k > 0.$$

Suppose there exists a path $x^i(t)$ $(0 \leqslant t \leqslant 1)$ joining R and S, any two points of $D(x_0)$, and lying entirely in $D(x_0)$. Assume $V(x)$ is such that $\dfrac{d^2V\{x(t)\}}{dt^2} > 0$ along the path $x^i(t)$ $(0 \leqslant t \leqslant 1)$. $V\{x(t)\}$ is a continuous function defined over a compact set; hence it attains its maximum on the set. Since $\dfrac{d^2V}{dt^2} > 0$ along the path, this maximum cannot occur for a value of t such that $0 < t < 1$, but must occur either at R or S. The path $x^i(t)$ from R to S thus lies inside any hypersurface of the family containing both R and S.

Consider $V\{x(t)\}$, where $x^i(t)$ $(0 \leqslant t \leqslant 1)$ is the path joining R to S. Then

$$\frac{d^2V}{dt^2} = \frac{\partial^2 V}{\partial x^i \partial x^j}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{\partial V}{\partial x^i}\frac{d^2x^i}{dt^2},$$

i.e.

$$\frac{d^2V}{dt^2} = \frac{\partial^2 V}{\partial x^i \partial x^j}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{\partial V}{\partial x^i}H^i(x, x'),$$

since $x^i = x^i(t)$ are the equations of a path. By the above result, if $V(x)$ is a function such that

$$\frac{\partial^2 V}{\partial x^i \partial x^j}\xi^i\xi^j + \frac{\partial V}{\partial x^i}H^i(\chi, \xi) > 0, \tag{3}$$

for all x inside of some hypersurface $V(x) = k$ and for all $\xi \neq 0$, then any path RS lying in $V(x) = k$ will lie within any $V(x) = k' < k$ in which both R and S lie. Since the left-hand side of (3) is positive homogeneous of the second degree in ξ we need only require that (3) hold for all vectors ξ for which $\sum_{i=1}^{n} \xi^i\xi^i = 1$, instead of for all ξ. In particular it is easily seen that

$$V(x) \equiv \sum_{i=1}^{n} (x^i - x_0^i)^2$$

will satisfy (3) inside of $\sum_{i=1}^{n}(x^i-x_0^i)^2 = k$ for k sufficiently small. In fact, the first term of (3), for this $V(x)$, is $2\sum_{i=1}^{n}\xi^i\xi^i = 2$ for $\sum_{i=1}^{n}\xi^i\xi^i = 1$. Since $\frac{\partial V}{\partial x^i} = 2(x^i-x_0^i)$ and since $H^i(x,\xi)$ is bounded for x bounded and for all ξ such that $\sum_{i=1}^{n}\xi^i\xi^i = 1$, the second term may be made less than the first by taking k sufficiently small.

Let $V(x) = k_1$ be a surface which lies in $D(x_0)$ and within which (3) holds. By the lemma there exists a neighbourhood $U(x_0)$ which is coverable by a spherical domain of normal coordinates $N(x)$ with origin at x in $U(x_0)$ and radius ρ (independent of x). Take $U(x_0)$, and with it ρ, so small that $N(x)$ lies in $V(x) = k_1$ for any origin x in $U(x_0)$. Take a $V(x) = k_2 < k_1$ which lies in $U(x_0)$, and let R and S be any two points within $V(x) = k_2$. Then $N(R)$, the spherical domain of normal coordinates with origin at R, contains $V(x) = k_2$ and is contained in $V(x) = k_1$. Now there exists a unique path RS joining R to S and lying in $N(R)$, and hence this path lies within $V(x) = k_1$. Since (3) holds within $V(x) = k_1$, the path RS lies in any hypersurface of the family containing both R and S, i.e., the path RS lies in $V(x) = k_2$ and thus the interior of $V(x) = k_2$ is convex. Since it is the only path joining R to S which lies in $N(R)$, and since $N(R)$ contains $V(x) = k_2$, it is the only path lying in $V(x) = k_2$ and joining R to S. Hence the interior of $V(x) = k_2$ is a simple convex neighbourhood of x_0, and Theorem 1 is proved.

Remark. It might be noted that Theorem 1 includes the case of extremals in a metric space. Since the path RS above lies in a spherical domain of normal coordinates, we see* that in this case we have a simple convex neighbourhood not merely with respect to extremals but even with respect to minimizing arcs.

3. A second application of the lemma is to the proof of

THEOREM 2. *Let M be a compact sub-set of a general space of paths S which is covered by a fixed set of coordinate systems. Then there exists a positive K, dependent on the choice of coordinate systems, such that a domain of normal coordinates (y) exists with origin at any point P of S and containing all y such that $\sum_{i=1}^{n} y^i y^i < K$.*

* Cf. the forthcoming paper by Mayer mentioned above.

Suppose this untrue. Then there exists a sequence of numbers $\rho_m \to 0$ and a sequence of points P_m of M such that it is impossible to set up a spherical domain of normal coordinates with origin at the mth point P_m and with radius greater than ρ_m. Since M is compact, a subsequence P'_m of P_m exists which converges to a limit point P of M. But this contradicts the lemma, which states that there is a neighbourhood $U(P)$ of the point P and a fixed positive ρ such that a spherical domain of normal coordinates exists with radius ρ, and with origin at any point of $U(P)$.

It might be noted that the results of this paper are valid for any metric space which is such that extremals in it are given as solutions of equations of the form of equations (1).

ON WHITTAKER'S ELECTROMAGNETIC 'SCALAR POTENTIALS'

By H. S. RUSE (*Southampton*)

[Received 22 April 1937]

THIS paper contains a discussion, in the light of special relativity, of a remarkable result obtained by E. T. Whittaker† in 1904 on the basis of the classical electromagnetic theory. His theorem is given in § 1, and a more detailed statement of the purpose of this paper appears in § 2.

1. The classical theorem

Suppose that at time $\bar{t}$ a particle of charge e having any prescribed motion is at the point P, $(\bar{x}, \bar{y}, \bar{z})$, with respect to a rectangular cartesian coordinate-system, and at that instant emits radiation which reaches an observer O, (x, y, z), at time t. Then

$$\bar{t} = t - r/c, \tag{1.1}$$

where $$r = \{(x-\bar{x})^2+(y-\bar{y})^2+(z-\bar{z})^2\}^{\frac{1}{2}}.$$

Let $\mathbf{v} \equiv (v_x, v_y, v_z) \equiv (d\bar{x}/d\bar{t}, \ldots, \ldots)$ be the velocity of the particle at the instant $\bar{t}$, and let $\mathbf{r}$ denote the 3-vector $\overrightarrow{PO}$ of components $(x-\bar{x}, y-\bar{y}, z-\bar{z})$. Then, as is well known,‡ the potential at O due to the moving particle is given in suitable units by

$$\mathbf{a} = \frac{e\mathbf{v}}{cr-\mathbf{r}\cdot\mathbf{v}}, \qquad \phi = \frac{ec}{cr-\mathbf{r}\cdot\mathbf{v}}, \tag{1.2}$$

where $\mathbf{r}\cdot\mathbf{v}$ is the scalar product of the vectors $\mathbf{r}$, $\mathbf{v}$, and where

$$\mathbf{a} \equiv (a_x, a_y, a_z) \equiv (\phi^1, \phi^2, \phi^3)$$

is the vector potential and $\phi \equiv \phi^4$ the scalar potential at O (time t). In the language of relativity, $\phi^i \equiv (\phi^1, \phi^2, \phi^3, \phi^4)$ are the components of a contravariant 4-vector of the galilean space-time. If the field is set up by a number of particles the potential is obtained by summing the expressions (1.2) over the whole system.

Whittaker's theorem may be stated as follows. If we write

$$\left.\begin{aligned} F &\equiv -e\,\mathrm{argtanh}\frac{z-\bar{z}}{t-\bar{t}} \\ G &\equiv e\arctan\frac{y-\bar{y}}{x-\bar{x}} \\ \psi &\equiv \tfrac{1}{2}e\log\{(x-\bar{x})^2+(y-\bar{y})^2\} \end{aligned}\right\}, \tag{1.3}$$

† E. T. Whittaker, *Proc. London Math. Soc.* (2), 1 (1904), 367–72.

‡ See, e.g., J. H. Jeans, *Mathematical Theory of Electricity and Magnetism* (Cambridge, 1923), 575.

then

$$\left.\begin{aligned} a_x \equiv \phi^1 &= -\frac{\partial G}{\partial y}+\frac{\partial \psi}{\partial x} \\ a_y \equiv \phi^2 &= \frac{\partial G}{\partial x}+\frac{\partial \psi}{\partial y} \\ a_z \equiv \phi^3 &= \frac{1}{c}\frac{\partial F}{\partial t}+\frac{\partial \psi}{\partial z} \\ \phi \equiv \phi^4 &= -\frac{\partial F}{\partial z}-\frac{1}{c}\frac{\partial \psi}{\partial t} \end{aligned}\right\}, \tag{1.4}$$

where, in performing the partial differentiations, we remember that the motion of the particle is prescribed, so that $\bar{x}$, $\bar{y}$, $\bar{z}$ are known functions of $\bar{t}$, and hence, by (1.1), of x, y, z, t. The form of the function F given by Whittaker is

$$F = e\,\mathrm{argsinh}\frac{\bar{z}-z}{\{(\bar{x}-x)^2+(\bar{y}-y)^2\}^{\frac{1}{2}}},$$

which, because of (1.1), is equivalent to that given in (1.3).

Not the least interesting feature of this result is that the electric and magnetic field-strengths

$$\mathbf{d} = -\frac{1}{c}\frac{\partial \mathbf{a}}{\partial t}-\mathrm{grad}\,\phi, \qquad \mathbf{h} = \mathrm{curl}\,\mathbf{a} \tag{1.5}$$

depend only upon the functions F and G, and not upon ψ; for example,

$$d_x = \frac{\partial^2 F}{\partial x \partial z}+\frac{1}{c}\frac{\partial^2 G}{\partial y \partial t}, \qquad h_x = \frac{1}{c}\frac{\partial^2 F}{\partial y \partial t}-\frac{\partial^2 G}{\partial x \partial z}.$$

Consequently *the field is determined by the two functions F and G.* They were called *scalar potential functions* by the discoverer, although the term 'scalar' was of course used in a pre-relativistic sense.

Hereafter we shall suppose the units of charge and velocity so chosen that $e = 1$ and $c = 1$.

2. Discussion

Perhaps the most noticeable feature of the formulae (1.3) for F, G, ψ and of equations (1.4) for $\mathbf{a}$, ϕ is their asymmetry; for x, y, z, t appear in pairs, x going with y and z with t, in a manner reminding one strongly of Dirac's equations. It is therefore a matter of some interest to investigate the invariantive nature of these formulae, and to find whether they can be expressed in terms of ordinary

tensors without the introduction of some specially devised calculus like that of spinors. It is to this investigation that the remainder of this paper is devoted. It is shown that the formulae are in fact expressible in terms of ordinary tensors, a result which, though perhaps not remarkable in itself, is interesting because of the unusual form of the tensorized equations.

We begin by observing that a set of functions F, G, ψ belongs to every galilean coordinate-system $(x) \equiv (x, y, z, t)$; that is, we have, in relation to any system $(x') \equiv (x', y', z', t')$ obtained by applying a Lorentz transformation to the system (x), the set of relations

$$a'_x = -\frac{\partial G'}{\partial y'} + \frac{\partial \psi'}{\partial x'}, \text{ etc.}, \qquad \phi' = -\frac{\partial F'}{\partial z'} - \frac{\partial \psi'}{\partial t'},$$

in which F', G', ψ' *are the same functions of* (x'), $(\bar{x}')$ *as* F, G, ψ *are of* (x), $(\bar{x})$, and $\mathbf{a}'$, ϕ' are the components of the vector and scalar potential in the new system. Now F', G', ψ' will obviously depend in some way upon F, G, ψ and upon the Lorentz transformation, but it does not follow, and is indeed untrue, that F', G', ψ' are the respective transforms of F, G, ψ when the latter are treated as scalars; that is, it is not in general true that F', G', ψ' are respectively equal to F, G, ψ. Therefore, we might conclude, F, G, ψ cannot be scalars in the sense of tensor analysis.

But to reach this conclusion at the present stage is premature, for the three functions do in fact appear as scalars in the final tensorized form of the theorem. This, however, does not mean that they retain their functional form under a Lorentz transformation, but merely that, given F, G, ψ in any one coordinate-system (x), there exists in any other system (x') a set of functions $'F$, $'G$, $'\psi$ which are obtained from F, G, ψ by the simple substitution of the x's in terms of the x''s, and which play a part similar to that played by F, G, ψ in the old system. This elementary point is laboured because it is essential that the term 'scalar', as used in this paper, should not be taken to imply the preservation of functional form as well as the numerical invariance that is the essential property of scalars.

The matter becomes clearer when we make our first tentative steps towards the generalization of Whittaker's theorem. Returning to the classical point of view, let $\mathbf{f}$, $\mathbf{g}$ denote the 3-vectors of components

$$\mathbf{f} \equiv (0, 0, F), \qquad \mathbf{g} \equiv (0, 0, G), \tag{2.1}$$

both of which are parallel to the axis of z. Then (1.4) may be written†

$$\left.\begin{aligned} \mathbf{a} &= -\operatorname{curl}\mathbf{g}+\frac{\partial \mathbf{f}}{\partial t}+\operatorname{grad}\psi \\ \phi &= -\operatorname{div}\mathbf{f}-\frac{\partial \psi}{\partial t} \end{aligned}\right\}. \tag{2.2}$$

Being vector equations, these remain formally unaltered by a change of the (x, y, z)-axes to a new rectangular set. But relative to the new set, $\mathbf{f}$ and $\mathbf{g}$ have components (f_x, f_y, f_z), (g_x, g_y, g_z) with f_x, f_y, g_x, g_y in general non-zero; that is, they are parallel to one another but not to the new z-axis. Now, since the original z-axis may be regarded as having an arbitrary direction relative to the new system, it follows that in any coordinate-system there is a pair of parallel vectors $\mathbf{f}, \mathbf{g}$ and a corresponding function ψ associated with every spatial direction, and that equations of the form (2.2) hold for each choice of $\mathbf{f}, \mathbf{g}, \psi$. The original formulae (1.3), (1.4) of Whittaker are merely those appropriate to the case when $\mathbf{f}$ and $\mathbf{g}$ are both parallel to the axis of z.

The appearance of six non-zero components of the two 3-vectors $\mathbf{f}, \mathbf{g}$ suggests that the tensor formulation of the result may involve a 6-vector. This is actually the case, as we now proceed to show.

3. The Hertzian tensor

Let $$ds^2 = g_{ij}\,dx^i dx^j \quad (i,j = 1,2,3,4)$$
define the metric of space-time, for the moment not necessarily flat. The electromagnetic field-equations *in vacuo* may be written

$$(F^{ij})_j = 0, \tag{3.1}$$

$$F_{ij} = \frac{\partial \phi_i}{\partial x^j}-\frac{\partial \phi_j}{\partial x^i}, \tag{3.2}$$

$$(\phi^i)_i = 0, \tag{3.3}$$

where F^{ij} is the 6-vector of field strength, ϕ_i is the potential 4-vector, and suffixes outside brackets denote covariant differentiations. In the case of a flat space-time with galilean coordinates (x, y, z, t) we have

$$ds^2 = -dx^2-dy^2-dz^2+dt^2, \tag{3.4}$$

$$F^{ij} \equiv \begin{pmatrix} 0 & h_z & -h_y & -d_x \\ -h_z & 0 & h_x & -d_y \\ h_y & -h_x & 0 & -d_z \\ d_x & d_y & d_z & 0 \end{pmatrix}, \tag{3.5}$$

† Vector formulae for the field-strengths $\mathbf{d}, \mathbf{h}$ in terms of $\mathbf{f}, \mathbf{g}$ are given by Whittaker, loc. cit.

(i specifying the row and j the column), and

$$(\phi^1, \phi^2, \phi^3, \phi^4) = (-\phi_1, -\phi_2, -\phi_3, \phi_4) = (a_x, a_y, a_z, \phi). \tag{3.6}$$

By substituting from (3.3) in (3.2) and using (3.4) we get the well-known† wave equation

$$(\phi^i)^j_j + R^{ij}\phi_j = 0, \tag{3.7}$$

R^{ij} being the contravariant form of the contracted Riemann-Christoffel tensor. Now the divergence-equation (3.3) may be satisfied identically by putting

$$\phi^i = (Z^{ij})_j - g^{ij}\frac{\partial \psi}{\partial x^j}, \tag{3.8}$$

where Z^{ij} is an arbitrary 6-vector and ψ any scalar that satisfies the partial differential equation

$$\Delta_2 \psi \equiv g^{ij}(\psi)_{ij} = 0. \tag{3.9}$$

This is so because $(Z^{ij})_{ji} \equiv 0$, as is easily proved. Substituting from (3.8) in (3.7), we find, after some reduction, that Z^{ij} must satisfy

$$\{(Z^{ij})^k_k + R^{ij}_{\cdot\cdot kl}\, Z^{kl} + R^j_{\cdot k}\, Z^{ik} + R^i_{\cdot k}\, Z^{kj}\}_j = 0. \tag{3.10}$$

The Riemann-Christoffel tensor $R^{ij}_{\cdot\cdot kl}$ and the contracted tensor $R^j_{\cdot k}$ are of course zero in a flat space-time.

Z^{ij} is the *Hertzian tensor*‡ which arises out of the classical theory of the Hertzian oscillator. When the metric has the galilean form (3.4), the equations (3.8), with $\psi \equiv 0$ and all the components of Z^{ij} zero except $Z^{34} = -Z^{43} = \Pi$, say, give

$$a_x = 0, \qquad a_y = 0, \qquad a_z = \frac{\partial \Pi}{\partial t}, \qquad \phi = -\frac{\partial \Pi}{\partial z},$$

whence

$$d_x = \frac{\partial^2 \Pi}{\partial x \partial z}, \ldots, \ldots, \qquad h_x = \frac{\partial^2 \Pi}{\partial y \partial t}, \ldots, \ldots,$$

so that Π is the potential function customarily introduced§ in the theory of the Hertzian doublet with axis along Oz.

Now consider the form of (3.8) in the galilean case (3.4). If we write

$$Z^{ij} \equiv \begin{pmatrix} 0 & -g_z & g_y & f_x \\ g_z & 0 & -g_x & f_y \\ -g_y & g_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix} \tag{3.11}$$

† A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge, 1924), 175.

‡ W. Gordon, *Ann. der Physik*, 72 (1923), 443 et seq. Cf. also J. Frenkel, *Lehrbuch der Elektrodynamik*, i (Berlin, 1926), 253. Equations (3.10) above may be compared with Gordon's formulae (128), (113).

§ See, e.g., R. A. Houston, *Introduction to Mathematical Physics* (London, 1912), 151.

and remember that covariant derivatives are now ordinary partial derivatives, the equations (3.8), in conjunction with (3.6), become precisely (2.2), namely

$$\mathbf{a} = -\operatorname{curl}\mathbf{g} + \frac{\partial \mathbf{f}}{\partial t} + \operatorname{grad}\psi,$$

$$\phi = -\operatorname{div}\mathbf{f} - \frac{\partial \psi}{\partial t}.$$

If further the coordinate-system is chosen so that the f's and g's are all zero except $f_z \equiv F$ and $g_z \equiv G$, say, we have

$$Z^{ij} \equiv \begin{pmatrix} 0 & -G & 0 & 0 \\ G & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & -F & 0 \end{pmatrix}, \tag{3.12}$$

and (3.8) reduce to the equations (1.4) from which we started.

The functions $\pm F$, $\pm G$ thus appear as components of a 6-vector of which all the other components are zero. This result may seem to contradict the assertion made in § 2 that F and G are scalars, but their scalar nature will be apparent when we have investigated more closely the actual form of the tensor Z^{ij} appropriate to the theory of a single charged particle.

4. The principal null directions of the Hertzian tensor and the associated ennuples

(The work of the present section holds for any space-time, whether flat or not.)

Let Λ be a principal scalar of the tensor Z^{ij}, so that

$$\det|Z_{ij} + \Lambda g_{ij}| = 0. \tag{4.1}$$

Then if the four roots of this equation in Λ are distinct, as we shall suppose them to be, two are real and equal in magnitude but opposite in sign, while the other two are conjugate pure imaginaries;† we assume that none of them is zero. Let $\pm\lambda$ be the real roots and $\pm i\mu$ be the imaginary roots, so that λ and μ are both real, and let ξ^i, η^i, ζ^i, ζ^{i*} be four vectors defining the principal directions of the

† See H. S. Ruse, *Proc. London Math. Soc.* 41 (1936), 308, 310. The proof there given is for the electromagnetic 6-vector F_{ij}, but holds for any real 6-vector.

tensor Z^{ij}. Then

$$\left.\begin{aligned}(Z_{ij}+\lambda g_{ij})\xi^j &= 0 \quad (a)\\ (Z_{ij}-\lambda g_{ij})\eta^j &= 0 \quad (b)\\ (Z_{ij}+i\mu g_{ij})\zeta^j &= 0 \quad (c)\\ (Z_{ij}-i\mu g_{ij})\zeta^{i*} &= 0 \quad (d)\end{aligned}\right\}. \tag{4.2}$$

It is evident that only the ratios of the four vectors can be determined from these equations, that is, their components are in each case indeterminate to the extent of a scalar factor. But, since the ratios of the ξ's and η's are evidently real, we may suppose their factors chosen so that the actual components are real. Likewise it is obvious from (4.2) (c) and (d) that we may suppose the respective components of ζ^i and ζ^{i*} to be complex conjugates of one another.

Let $(\xi\eta)$ denote the inner product $\xi_i\eta^i$ of the vectors ξ_i, η^i, so that

$$(\xi\eta) = (\eta\xi) = g_{ij}\xi^i\eta^j,$$

and so for any pair of vectors. Then, multiplying (4.2) (a) by ξ^j, summing for j and using the fact that Z^{ij} is skew, we get $\lambda(\xi\xi) = 0$, whence $(\xi\xi) = 0$ because we are assuming $\lambda \neq 0$. That is, ξ^i is a null vector, and similarly η^i, ζ^i, ζ^{i*} are null. Thus

$$(\xi\xi) = 0 = (\eta\eta) = (\zeta\zeta) = (\zeta^*\zeta^*). \tag{4.3}$$

It is also easy to prove that

$$(\xi\zeta) = 0 = (\xi\zeta^*) = (\eta\zeta) = (\eta\zeta^*), \tag{4.4}$$

and that† $\qquad (\xi\eta) \neq 0, \qquad (\zeta\zeta^*) \neq 0.$

Each of the null vectors is therefore perpendicular to itself and to two of the others, but not to the third. We suppose the scalar factors of ξ^i, η^i chosen so as to make

$$(\xi\eta) > 0, \tag{4.5}$$

and remark that it can be proved either by matrix methods or by the theory of spinors‡ that the inequality

$$(\zeta\zeta^*) < 0 \tag{4.6}$$

necessarily holds; that $(\zeta\zeta^*)$ is *real* is obvious because ζ^{i*} is the complex conjugate of ζ^i.

† Ruse, loc. cit., 307.

‡ H. S. Ruse, *Proc. R. S. Edinburgh*, 57 (1936–7), 117, equation (8.11). In that paper the vectors ξ, η, ζ, ζ^* bear the same relation to the 6-vector of electric and magnetic moment (ibid., § 11) as the present vectors bear to the Hertzian tensor. The inequality (4.6) may also be proved directly by reducing the matrices Z_{ij}, g_{ij} to canonical forms, a fact which I have verified with the kind assistance of Dr. A. C. Aitken.

It can further be shown by geometrical† or analytical‡ methods that

$$Z^{ij} = -\frac{\lambda}{(\xi\eta)}(\xi^i\eta^j - \eta^i\xi^j) - \frac{\mu i}{(\zeta\zeta^*)}(\zeta^i\zeta^{j*} - \zeta^{i*}\zeta^j). \tag{4.7}$$

Now let α, σ be arbitrary real scalars, and write

$$\left.\begin{aligned} p^j &= \frac{i(e^{i\alpha}\zeta^j + e^{-i\alpha}\zeta^{j*})}{(\zeta\zeta^*)^{\frac{1}{2}}\sqrt{2}}, & q^j &= \frac{e^{i\alpha}\zeta^j - e^{-i\alpha}\zeta^{j*}}{(\zeta\zeta^*)^{\frac{1}{2}}\sqrt{2}} \\ r^j &= \frac{\sigma\xi^j - \sigma^{-1}\eta^j}{(\xi\eta)^{\frac{1}{2}}\sqrt{2}}, & s^j &= \frac{\sigma\xi^j + \sigma^{-1}\eta^j}{(\xi\eta)^{\frac{1}{2}}\sqrt{2}} \end{aligned}\right\}. \tag{4.8}$$

Then it follows at once from (4.3), (4.4), (4.5), and (4.6) that p^j, q^j, r^j, s^j are four real mutually orthogonal unit vectors constituting an orthogonal ennuple, the first three being space-like and the last time-like; thus

$$(pp) = -1 = (qq) = (rr), \qquad (ss) = 1,$$

and (pq), (pr), (ps), (qr), (qs), (rs) are all zero. Also from (4.8) it follows that

$$\begin{aligned} p^iq^j - q^ip^j &= -i(\zeta^i\zeta^{j*} - \zeta^{i*}\zeta^j)/(\zeta\zeta^*), \\ r^is^j - s^ir^j &= (\xi^i\eta^j - \eta^i\xi^j)/(\xi\eta), \end{aligned}$$

so that, by (4.7),§

$$Z^{ij} = -\lambda(r^is^j - s^ir^j) + \mu(p^iq^j - q^ip^j). \tag{4.9}$$

Suppose now that the tensor Z^{ij} at the world-point (x^i) is referred to the local galilean system of axes defined by the ennuple (p, q, r, s). If we use a, b, c for ennuplet (scalar) suffixes and write

$$\left.\begin{aligned} h^i_a &\equiv (h^i_1, \ldots, h^i_4) \equiv (p^i, q^i, r^i, s^i) \\ h^a_i &\equiv (h^1_i, \ldots, h^4_i) \equiv (-p_i, -q_i, -r_i, s_i) \end{aligned}\right\}, \tag{4.10}$$

so that

$$h^a_i h^i_b = \delta^a_b, \tag{4.11}$$

then the ennuplet (galilean) components of Z^{ij} at (x^i) are

$$\begin{aligned} Z^{ab} &= Z^{ij}h^a_i h^b_j \\ &= \{-\lambda(h^i_3 h^j_4 - h^i_4 h^j_3) + \mu(h^i_1 h^j_2 - h^i_2 h^j_2)\}h^a_i h^b_j \quad \text{by (4.9) and (4.10)} \\ &= -\lambda(\delta^a_3\delta^b_4 - \delta^a_4\delta^b_3) + \mu(\delta^a_1\delta^b_2 - \delta^a_2\delta^b_1) \quad \text{by (4.11)}, \end{aligned}$$

that is,

$$Z^{ab} = \begin{pmatrix} 0 & \mu & 0 & 0 \\ -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & \lambda & 0 \end{pmatrix}.$$

† H. S. Ruse, *Proc. London Math. Soc.* 41 (1936), 309 and 310, (4.16).

‡ J. L. Synge, *Univ. of Toronto Studies*, Applied Math. Series, No. 1 (1935), 44, (9.15).

§ Cf. J. A. Schouten, *Der Ricci-Kalkül* (Berlin, 1924), 49, equation (132).

If this is compared with (3.12), namely

$$Z^{ij} = \begin{pmatrix} 0 & -G & 0 & 0 \\ G & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & -F & 0 \end{pmatrix},$$

it is evident that the two matrices are the same, provided that $\lambda = -F$, $\mu = -G$.

It now appears, therefore, that $\pm F$ and $\pm iG$ are the principal scalars of the Hertzian tensor, and we are at last in a position to give a fully tensorized version of Whittaker's theorem.

5. The 4-potential of a charged particle

Let the metric of space-time, which we assume hereafter to be flat, be given in terms of any real coordinates x^i by the usual formula

$$ds^2 = g_{ij}\,dx^i dx^j. \tag{5.1}$$

When the coordinates are galilean this has the form

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2. \tag{5.2}$$

Suppose that the 'past' sheet of the null cone of an observer O at the world-point (x^i) intersects the world-line of a particle of unit charge at P, $(\bar{x}^i)$, so that the $\bar{x}^i$ are the coordinates of the particle at the instant when radiation left it to reach the observer at (x^i). If the $\bar{x}^i$ are expressed as functions of the proper-time τ of the particle, we have

$$\bar{x}^i = \bar{x}^i(\tau) \tag{5.3}$$

and

$$d\tau^2 = \bar{g}_{ij}\,d\bar{x}^i d\bar{x}^j. \tag{5.4}$$

Let s be the geodesic (rectilinear 4-dimensional) distance PO, and write

$$\Omega = \tfrac{1}{2}s^2. \tag{5.5}$$

Then Ω is a scalar function of the x's and $\bar{x}$'s, and the fact that the points P, O are on the same null geodesic may be written

$$\Omega = 0. \tag{5.6}$$

Substitution from (5.3) in (5.6) gives an equation for τ which may be solved to give τ as a function of the x^i, say

$$\tau = \tau(x^i). \tag{5.7}$$

The equations (5.3), (5.7) together give the $\bar{x}$'s as functions of the x's.

Let

$$\Omega_{i|} \equiv \frac{\partial \Omega}{\partial x^i}, \tag{5.8}$$

where the vertical stroke following the suffix indicates that the partial

differentiation with respect to x^i is in the strict sense, i.e., with the $\bar{x}$'s kept constant as well as the other x's. The proper partial derivative is

$$\Omega_i = \Omega_{i|} + \frac{\partial \Omega}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^i},$$

and is zero by (5.6). We note in passing that, when the coordinates are galilean,

$$\Omega = \tfrac{1}{2}\{-(x-\bar{x})^2-(y-\bar{y})^2-(z-\bar{z})^2+(t-\bar{t})^2\}, \tag{5.5}'$$

$$\Omega_{1|} = -(x-\bar{x}), \qquad \Omega_{2|} = -(y-\bar{y}), \qquad \Omega_{3|} = -(z-\bar{z}), \qquad \Omega_{4|} = t-\bar{t}. \tag{5.8}'$$

Now let (p^i, q^i, r^i, s^i) be a real orthogonal ennuple at the observer's world-point (x^i) such that

$$(p^i)_j = 0 = (q^i)_j = (r^i)_j = (s^i)_j, \tag{5.9}$$

the four vectors being arbitrary but for this restriction, which may be imposed because the Riemann-Christoffel tensor is zero in a flat space-time and equations (5.9) are therefore completely integrable. In general the components of p^i, q^i, r^i, s^i are functions of the x^i, but in a galilean system they are all constants because the covariant derivatives in (5.9) are then ordinary partial derivatives.

Further, write

$$\left.\begin{aligned} F &\equiv \operatorname{argtanh}(r^k\Omega_{k|}/s^l\Omega_{l|}) \\ G &\equiv \arctan(q^k\Omega_{k|}/p^l\Omega_{l|}) \\ \psi &\equiv \tfrac{1}{2}\log\{(p^k\Omega_{k|})^2+(q^k\Omega_{k|})^2\} \end{aligned}\right\}, \tag{5.10}$$

where the index 2 in the third formula denotes the square. It is evident that F, G, ψ are all scalars for transformations of coordinates.

The tensor form of Whittaker's theorem may now be stated as follows:

The 4-potential ϕ^i of the particle at the world-point $P,(\bar{x}^i)$, as measured by an observer O at (x^i) on the null cone of P, is

$$\phi^i = (Z^{ij})_j - g^{ij}\frac{\partial \psi}{\partial x^j}, \tag{5.11}$$

where

$$Z^{ij} = F(r^is^j-s^ir^j)-G(p^iq^j-q^ip^j). \tag{5.12}$$

The partial derivatives of Z^{ij} and ψ in (5.11) are to be understood in the proper sense, that is, with the $\bar{x}$'s treated as functions of the x's. The only strict partial derivative appearing in the theorem is in fact the one mentioned above, namely $\Omega_{i|}$.

The theorem can be proved directly by substituting from (5.12)

in (5.11), using (5.9), and performing the covariant differentiations required by (5.11). In the course of the proof, which we omit because it is straightforward but long and uninteresting, it is necessary to use the fact that Ω satisfies† the partial differential equations

$$g^{ij}\Omega_{i|}\Omega_{j|} = 2\Omega, \qquad \Omega_{i|j|} = g_{ij}. \tag{5.13}$$

In this way we obtain ultimately

$$\phi_i = \frac{\partial}{\partial x^i}\left(\log\frac{\partial\Omega}{\partial\tau}\right), \tag{5.14}$$

where the partial differentiation with respect to x^i is strict,‡ and this is the correct formula for the potential of a particle of unit charge in a flat space-time.§

6. Comments

Consider the case when the coordinates are galilean and the vectors (p^i, q^i, r^i, s^i) are chosen parallel to the axes of x, y, z, t respectively. Then

$$\left.\begin{aligned} p^i &= (1,0,0,0) = \delta^i_1 \\ q^i &= (0,1,0,0) = \delta^i_2 \\ r^i &= (0,0,1,0) = \delta^i_3 \\ s^i &= (0,0,0,1) = \delta^i_4 \end{aligned}\right\}, \tag{6.1}$$

so that, by (5.8)′,

$$p^k\Omega_{k|} = -(x-\bar{x}), \quad \ldots, \quad \ldots, \quad s^k\Omega_{k|} = t-\bar{t},$$

and consequently

$$\left.\begin{aligned} F &= -\operatorname{argtanh}\frac{z-\bar{z}}{t-\bar{t}}, \qquad G = \arctan\frac{y-\bar{y}}{x-\bar{x}} \\ \psi &= \tfrac{1}{2}\log\{(x-\bar{x})^2+(y-\bar{y})^2\} \end{aligned}\right\}. \tag{6.2}$$

It follows at once that the general theorem of § 5 reduces to the classical theorem from which we started.

When the vectors of the ennuple are not taken parallel to the coordinate-axes the functions F, G, ψ have a less simple form than in (6.2). In general they involve six constant parameters correspond-

† H. S. Ruse, *Proc. London Math. Soc.* 31 (1930), 225, 226, where the function represented by Ω is twice that of the present paper.

‡ If the partial differentiation were not strict, (5.14) would be obviously wrong because ϕ_i would then be the gradient of the scalar $\partial\Omega/\partial\tau$.

§ H. S. Ruse, *Quart. J. of Math.* (Oxford), 1 (1930), 154.

ing to the six degrees of freedom of the ennuple. To take an example, suppose that the vectors p^i, q^i in (6.1) are rotated through an angle α about the plane of r^i, s^i. We obtain

$$p^i = (\cos\alpha, \sin\alpha, 0, 0),$$
$$q^i = (-\sin\alpha, \cos\alpha, 0, 0),$$
$$r^i = \delta^i_3, \qquad s^i = \delta^i_4,$$

which involve the single constant α. Substitution of these values in the general formula give F, G, ψ, Z^{ij} in a less simple form than in the original theorem, but not, of course, in their most general form.

The possibility of rotating the ennuple as well as transforming the coordinates explains the curious circumstance, noted in § 2, that in any galilean system there is a pair of parallel 3-vectors $\mathbf{f}$, $\mathbf{g}$ associated with every spatial direction. The functions F, G, ψ are scalars for transformations of coordinates, but have a more elaborate law of transformation for rotations of the ennuple. It is now evident that they preserve their functional form under a Lorentz transformation, i.e., under a rotation of the coordinate-axes, if and only if the ennuple is also rotated so that its orientation relative to the rotated axes is the same as its original orientation relative to the unrotated axes. In the classical theorem of Whittaker the arms of the ennuple are parallel to the coordinate-axes, and must be kept so if the functional form of F, G, ψ is to be preserved under Lorentz transformations.

Had the unproved general theorem been stated without the preliminary work of §§ 2–4, its dependence upon the largely arbitrary selection of an orthogonal ennuple might have caused surprise or even scepticism. But once it is established that Whittaker's formula for the 4-potential involves the Hertzian 6-vector, the appearance of an ennuple may be expected because, as we have seen, the introduction of a general 6-vector into space-time requires the explicit or implicit specification of a set of four null vectors, or, equivalently,† of an orthogonal ennuple; and, as there is manifestly no such set of null vectors intrinsically connected with a physical configuration consisting of a single particle and a single observer, it may also be foreseen that there will be a considerable degree of arbitrariness in

† Equivalently in the sense that to be given the ennuple is to be given the directions of the null vectors, which, by (4.8), are proportional to $r^j \pm s^j$, $p^j \pm iq^j$.

the selection of the ennuple. That the Hertzian tensor appears at all in this connexion is of some interest, especially if it be remembered that one classical method of obtaining the 4-potential of a moving point-charge is to reduce the problem to that of finding the 4-potential of a stationary doublet plus a stationary point-charge.†

† Cf. J. Larmor, *Phil. Mag.* 44 (1897), 508, § 9 [reprinted in Larmor's *Mathematical and Physical Papers*, ii (Cambridge, 1929), 145, and in his *Aether and Matter* (Cambridge, 1900), 221].